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RESEARCHES OF THE BOUNDEDNESS AND STABILITY OF THE SOLUTIONS OF NON-LINEAR DIFFERENTIAL EQUATIONS

By
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(*Presented by* P. TURÁN)

Introduction

The purpose of this paper—arisen by a strong influence of a book of R. BELLMAN[1] — is the investigation indicated in the title, considering different types of equations. The results attained in this topic by LIAPUNOV, POINCARÉ and PERRON[2] relate to equations of the form[3] $\frac{d\mathbf{z}}{dt}=\mathbf{A}(t)\mathbf{z}+\boldsymbol{\varphi}(\mathbf{z},t)$ and involve, concerning the term $\boldsymbol{\varphi}(\mathbf{z},t)$, the requirement $\frac{\|\boldsymbol{\varphi}(\mathbf{z},t)\|}{\|\mathbf{z}\|}\to 0$ $(t\to+\infty, \|\mathbf{z}\|\to 0)$ without detailing the dependence on t. In linear domain there are theorems[4] prescribing less restrictive conditions. These theorems, assuming $\boldsymbol{\varphi}(\mathbf{z},t)$ in the form $\mathbf{B}(t)\mathbf{z}$, postulate $\int^{\infty}\|\mathbf{B}(t)\|dt<\infty$ or $\|\mathbf{B}(t)\|<b$, bounding in this manner the value of $\|\mathbf{B}\|$.

The present paper will give for non-linear equations analogous, less restrictive conditions relative to the dependence on t and $\mathbf{z}$, moreover also conditions connecting the growth of $\boldsymbol{\varphi}(\mathbf{z},t)$ by t with that by $\mathbf{z}$. On the other hand, $\boldsymbol{\varphi}(\mathbf{z},\mathrm{t})$ will be assumed in the form $\mathbf{B}(t)\boldsymbol{\varphi}(\mathbf{z})$.

In our discussion the following lemma[5] will play an important role:

LEMMA. Let the functions $Y(x)$ and $F(x)$ be continuous, $F(x)\geqq 0$ in $x_0\leqq x\leqq X_0$ and $k>0$. Suppose the function $\omega(u)$ to be continuous, non-decreasing and $\omega(u)>0$ for $u>0$. Denote the integral $\int_{u_0}^{u}\frac{dt}{\omega(t)}$ $(u_0>0, u\geqq 0)$

[1] R. BELLMAN, *Stability theory of differential equations* (New York, 1953).

[2] O. PERRON, Über Stabilität und asymptotisches Verhalten der Integrale von Differentialgleichungssystemen, *Math. Zeitschrift*, **29** (1929), pp. 129—160.

[3] See the notations in paragraph 1.

[4] R. BELLMAN, The stability of solutions of linear differential equations, *Duke Math. Journal*, **10** (1943), pp. 643—647.

[5] I. BIHARI, A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations, *Acta Math. Acad. Sci. Hung.*, **7** (1956), pp. 83—94.

by $\Omega(u)$ and its inverse by $\Omega^{-1}(u)$ defined for $\Omega(0) \leqq u \leqq \Omega(\infty)$. This interval may be infinite from one or both sides, accordingly as integrals

$$\int_0^1 \frac{dt}{\omega(t)}, \quad \int_1^\infty \frac{dt}{\omega(t)} \text{ converge or not.}$$

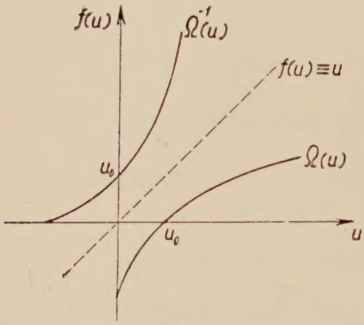

Fig. 1

The lemma states that the inequality

$$Y(x) \leqq k + \int_{x_0}^{x} F(t)\omega(Y(t))dt \qquad (x_0 \leqq x \leqq X_0)$$

implies the other inequality

$$Y(x) \leqq \Omega^{-1}\left(\Omega(k) + \int_{x_0}^{x} F(t)dt\right)$$

$$(x_0 \leqq x \leqq X_0' \leqq X_0)$$

where X_0' is determined by the requirement

$$\Omega(k) + \int_{x_0}^{X_0'} F(t)dt \leqq \Omega(\infty).$$

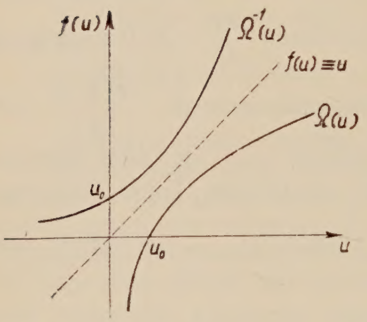

Fig. 2

If $\omega(u) \equiv u$, then the second inequality has the form

$$Y(x) \leqq k e^{\int_{x_0}^{x} F(t)dt}$$

and our lemma reduces itself to a lemma of Bellman.[6]

§ 1

Let us first consider the following vector-matrix equation (properly speaking: system of equations):

$$\text{(1)} \qquad \frac{d\mathbf{z}}{dt} = \mathbf{A}\mathbf{z} + \mathbf{B}(t)\boldsymbol{\varphi}(\mathbf{z}) \qquad (\mathbf{z}(0) = \mathbf{c})$$

where $\mathbf{A}$ means a constant (square) matrix, the vector $\mathbf{z}$ is a function of t, the vector $\boldsymbol{\varphi}(\mathbf{z})$ is a function of the components of $\mathbf{z}$ and the elements of the (square) matrix $\mathbf{B}(t)$ are functions of t. We investigate the equation for $t \geqq 0$ taking as initial value $\mathbf{z}(0) = \mathbf{c}$ where $\mathbf{c}$ is a constant vector.

Consider simultaneously the vector-matrix equation

$$\text{(2)} \qquad \frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y} \qquad (\mathbf{y}(0) = \mathbf{z}(0) = \mathbf{c})$$

[6] R. Bellman, loc. cit., p. 643.

and the matrix-matrix equation

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{y} \qquad (\mathbf{Y}(0) = \mathbf{I}) \tag{3}$$

where the parantheses indicate the corresponding initial values of the solutions and $\mathbf{I}$ is the identity matrix. As known, $\mathbf{y} = \mathbf{Y}\mathbf{c}$.

The norm of the column vector $\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ is defined by $\|\mathbf{y}\| = \sum_i |y_i|$ and the norm of the matrix $\mathbf{A} = (a_{ik})$ is defined by $\|\mathbf{A}\| = \sum_{i,k} |a_{ik}|$. These norms satisfy known inequalities.

The first fact we shall prove is expressed by

THEOREM 1. *If*

1. $\varphi(\mathbf{z})$ *is defined and continuous for* $\|\mathbf{z}\| \leqq K$ $(K > 0)$ *where* K *will be determined later;*

2. *all the solutions of the equation* (2) *are bounded for* $t \geqq 0$, *e.g. the real parts of the characteristic roots of* $\mathbf{A}$ *are negative or zero, and in this last case they are simple;*

3. $\mathbf{B}(t)$ *is continuous for* $t \geqq 0$ *and* $\int\limits_0^\infty \|\mathbf{B}(t)\|\, dt < \frac{1}{c_1} \int\limits_0^\infty \frac{dt}{\omega(t)}$ *where* $c_1 = \sup\limits_{t \geqq 0} \|\mathbf{Y}\|$ *and* $\omega(u)$ *is determined by* $\omega(u) \geqq \max\limits_{\|\zeta\| \leqq u} \|\varphi(\zeta)\|$;[7]

then all the solutions of (1) *exist and are bounded for* $t \geqq 0$, *provided that* $\|\mathbf{c}\|$ *is sufficiently small. The bound will be given explicitly.*

If $\omega(0) = 0$ (*i.e.* $\varphi(0) = 0$) *and* $\int\limits_0^1 \frac{dt}{\omega(t)}$ *is divergent, the solution* $\mathbf{z} = 0$ *is stable, provided that* $\|\mathbf{c}\|$ *is sufficiently small.*

Of course, if at least one of the integrals $\int\limits_0^1 \frac{dt}{\omega(t)}$, $\int\limits_1^\infty \frac{dt}{\omega(t)}$ is divergent, the second part of the condition 3 may be replaced by

$$\int\limits_0^\infty \|\mathbf{B}(t)\|\, dt < \infty.$$

[7] If $\omega(u)$ vanishes in some region of $u = 0$, then were quire $\int\limits_0^\infty \|\mathbf{B}(t)\|\, dt < \infty$. Assuming $\omega(u) \equiv \max\limits_{\|\zeta\| \leqq u} \|\varphi(\zeta)\|$, the function $\omega(u)$ is non-decreasing and satisfies the condition of the lemma. Taking $\omega(u) \geqq \max\limits_{\|\zeta\| \leqq u} \|\varphi(\zeta)\|$, we must suppose this explicitly.

If $\int\limits_1^\infty \frac{dt}{\omega(t)}$ is divergent, then the restriction "$\|\mathbf{c}\|$ sufficiently small" may be omitted here and in the following theorems too. If **A** has different characteristic roots, we can give explicitly a bound for c_1, viz. $c_1 \leqq \|\mathbf{T}^{-1}\|\cdot\|\mathbf{T}\|$ where **T** is the matrix of the characteristic vectors.

PROOF. As known, the solution of the *linear* inhomogeneous equation

$$\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u} + \mathbf{w}(t) \qquad (\mathbf{A} = \text{const}, \ \mathbf{u}(0) = \mathbf{c}) \tag{4}$$

can be written in the form

$$\mathbf{u} = \mathbf{y} + \int\limits_0^t \mathbf{Y}(t-t_1)\mathbf{w}(t_1)\,dt_1, \tag{5}$$

where $\mathbf{y}$ and $\mathbf{Y}$ are the solutions of (2) and (3), respectively.

Identifying the term $\mathbf{B}(t)\varphi(\mathbf{z})$ of the equation (1) with $\mathbf{w}(t)$ we get that every solution $\mathbf{z} = \mathbf{z}(t)$ of (1) satisfies the non-linear integral equation

$$\mathbf{z} = \mathbf{y} + \int\limits_0^t \mathbf{Y}(t-t_1)\mathbf{B}(t_1)\varphi(\mathbf{z}(t_1))\,dt_1 \tag{6}$$

and conversely. We make use of the inequalities satisfied by the norms

$$\|\mathbf{z}\| \leqq \|\mathbf{y}\| + \int\limits_0^t \|\mathbf{Y}(t-t_1)\|\,\|\mathbf{B}(t_1)\|\,\|\varphi(\mathbf{z}(t_1))\|\,dt_1 \qquad (t \geqq 0).$$

Being $\mathbf{y} = \mathbf{Y}\mathbf{c}$, we have $\|\mathbf{y}\| \leqq \|\mathbf{Y}\|\,\|\mathbf{c}\| \leqq c_1\|\mathbf{c}\|$, further $\|\varphi(\mathbf{z})\| \leqq \omega(\|\mathbf{z}\|)$. Therefore

$$\|\mathbf{z}\| \leqq c_1\|\mathbf{c}\| + c_1\int\limits_0^t \|\mathbf{B}(t_1)\|\,\omega(\|\mathbf{z}\|)\,dt_1 \qquad (t \geqq 0). \tag{6'}$$

Hence by the lemma

$$\|\mathbf{z}\| \leqq \Omega^{-1}\Big(\Omega(c_1\|\mathbf{c}\|) + c_1\int\limits_0^t \|\mathbf{B}(t_1)\|\,dt_1\Big) \qquad (t \geqq 0).^{8} \tag{7}$$

By virtue of the second part of the condition 3 the argument of Ω^{-1} in (7) is, for every positive t, within its definition domain, provided that $\|\mathbf{c}\|$

[8] If $\omega(u) = 0$ for $0 \leqq u \leqq \delta$, then assuming $\|\mathbf{c}\| < \delta$, $c_1\|\mathbf{c}\| < \delta$ and supposing that $\|\mathbf{z}\|$ attains the value δ for $t = t_0 > 0$ we get from (6′)

$$\delta \leqq c_1\|\mathbf{c}\| + 0 < \delta$$

what is a contradiction. Therefore $\|\mathbf{z}\| < \delta$ for $t \geqq 0$.

is sufficiently small. If $\int_1^\infty \frac{dt}{\omega(t)}$ is divergent, $\|\mathbf{c}\|$ need not be small at this end. By (7) the boundedness of $\|\mathbf{z}\|$ for $t \geqq 0$ is obvious.

In order to prove the stability of the solution $\mathbf{z}=0$ we argue as follows: choosing $\|\mathbf{c}\|$ small enough $\Omega(c_1\|\mathbf{c}\|)$ will be as small as wanted (it will approach $-\infty$ arbitrarily) and so the right-hand member of (7) will be for $t \geqq 0$ less than an arbitrary positive number, what is equivalent to the stated stability.

If e. g.[9] $\omega(u)=u^\alpha$ $(\alpha>1)$, then inequality (7) yields

$$\|\mathbf{z}\| \leqq \left[(c_1\|\mathbf{c}\|)^{1-\alpha}+(1-\alpha)c_1\int_0^t \|\mathbf{B}(t_1)\|dt_1\right]^{\frac{1}{1-\alpha}}. \tag{7'}$$

If $\|\mathbf{c}\| \to 0$, $(c_1\|\mathbf{c}\|)^{1-\alpha} \to +\infty$, and so the right-hand member of (7) tends uniformly to 0 for $t \geqq 0$. In order that the expression in (7′) may have a meaning for every $t \geqq 0$ and arbitrary $\alpha>1$ we need $\int_0^\infty \|\mathbf{B}(t_1)\|dt_1 < \frac{\|\mathbf{c}\|^{1-\alpha}}{(\alpha-1)c_1^\alpha}$.

This may be attained by assuming $\|\mathbf{c}\|$ sufficiently small and $\int_0^\infty \|\mathbf{B}(t)\|dt<\infty$.

In the present case $\int_0^1 \frac{dt}{\omega(t)}$ is divergent and $\int_1^\infty \frac{dt}{\omega(t)}$ is convergent.

If $\alpha<1$, then the bound given by (7′) is valid but we cannot assert the stability of $\mathbf{z}=0$ since $\int_0^1 \frac{dt}{\omega(t)}$ is convergent. The boundedness of $\|\mathbf{z}\|$ does not require $\|\mathbf{c}\|$ to be small.

REMARK 1. If we take the definition domain of $\varphi(\mathbf{z})$ a little wider than the bound K furnished by (7), then the existence of the solution of (1) for $t \geqq 0$ will be assured since the solution exists certainly in a region of $\mathbf{z}=0$ and can be continued up to the boundary of the domain $\|\mathbf{z}\| \leqq K+\varepsilon$ $(0 \leqq t < +\infty)$ where ε is small enough.

REMARK 2. Similar results may be obtained for the equation $\frac{d\mathbf{z}}{dt}=\mathbf{A}\mathbf{z}+\mathbf{f}(\mathbf{z},t)$ assuming $\|\mathbf{Y}\| \leqq L$, $\|\mathbf{f}(\mathbf{z},t)\| \leqq \omega(\|\mathbf{z}\|)\,g(t)$ and $\int^\infty g(t)dt<\infty$ $(g(t)>0)$ where $\omega(u)$ is a non-decreasing function and $\omega(0) \geqq 0$.

[9] The function $\omega(u)=u^\alpha$ is playing a part e.g. in the *scalar* equation $z'+z=b(t)z^\alpha$.

§ 2

THEOREM 2. *If we add to the hypotheses of Theorem 1 also the condition:*
4. *every solution of* (2) *tends to zero as* $t \to +\infty$ *(i. e.* **A** *has characteristic roots with negative real parts),*
then every solution of (1) *tends to zero as* $t \to +\infty$, *provided that* $\|\mathbf{c}\| = \|\mathbf{z}(0)\|$ *is sufficiently small.*[10]

PROOF. We can conclude from the explicit form of the solution of (2) that — provided $\|\mathbf{Y}\| \to 0$ as $t \to \infty$ — there exist positive constants a and c_1 such that

$$\|\mathbf{Y}\| \leqq c_1 e^{-at}, \quad \text{consequently} \quad \|\mathbf{y}\| \leqq c_1 \|\mathbf{c}\| e^{-at}.$$

Then we get from (6)

$$\|\mathbf{z}\| \leqq c_1 \|\mathbf{c}\| e^{-at} + c_1 \int_0^t e^{-a(t-t_1)} \|\mathbf{B}(t_1)\| \|\varphi(\mathbf{z}(t_1))\| dt_1. \tag{8}$$

We know by Theorem 1 that $\|\mathbf{z}\|$ is bounded — say $\|\mathbf{z}\| \leqq K$ — for sufficiently small $\|\mathbf{c}\|$, e. g. for $\|\mathbf{c}\| < \delta$. (Here K depends only on δ.) From (8)

$$\|\mathbf{z}\| \leqq c_1 \|\mathbf{c}\| e^{-at} + c_1 \int_0^{t/2} e^{-a(t-t_1)} \|\mathbf{B}(t_1)\| \omega(K) dt_1 + c_1 \int_{t/2}^{t} e^{-a(t-t_1)} \|\mathbf{B}(t_1)\| \omega(K) dt_1 \leqq$$

$$\leqq c_1 \|\mathbf{c}\| e^{-at} + c_1 \omega(K) e^{-a\frac{t}{2}} \int_0^{t/2} \|\mathbf{B}(t_1)\| dt_1 + c_1 \omega(K) \int_{t/2}^{t} \|\mathbf{B}(t_1)\| dt_1 \qquad (\|\mathbf{c}\| < \delta).$$

Being $\int_0^\infty \|\mathbf{B}(t)\| dt < \infty$, the right-hand member of the last formula tends to 0 as $t \to +\infty$. Consequently, $\|\mathbf{z}\| \to 0$ as $t \to +\infty$.

In linear case ($\varphi(\mathbf{z}) \equiv \mathbf{z}$) we can obtain more. From (8)

$$\|\mathbf{z}\| e^{at} \leqq c_1 \|\mathbf{c}\| + c_1 \int_0^t \|\mathbf{B}(t_1)\| e^{at_1} \|\mathbf{z}\| dt_1$$

and by the lemma

$$\|\mathbf{z}\| e^{at} \leqq c_1 \|\mathbf{c}\| e^{c_1 \int_0^t \|\mathbf{B}(t_1)\| dt_1} \qquad \text{and} \qquad \|\mathbf{z}\| \leqq c_1 \|\mathbf{c}\| e^{c_1 \int_0^t \|\mathbf{B}(t_1)\| dt_1 - at},$$

respectively, i. e. we can conclude, without any restriction on $\|\mathbf{c}\|$, that the solution $\mathbf{z}$ is bounded or tends to 0, if the exponent $c_1 \int_0^t \|\mathbf{B}(t_1)\| dt_1 - at$ remains bounded or tends to $-\infty$ (e. g. $\int_0^\infty \|\mathbf{B}(t) dt < \infty$), respectively.

[10] Originally more was assumed. J. CZIPSZER simplified the theorem and the proof too.

If, differently from the above assumption, we suppose $\|\mathbf{B}(t)\|$ to be bounded — say $\|\mathbf{B}(t)\| < b$ — then we get

$$\|\mathbf{z}\| \leq c_1 \|\mathbf{c}\| e^{(c_1 b - a)t}$$

and so, provided $b < \frac{a}{c_1}$, $\|\mathbf{z}\| \to 0$ as $t \to +\infty$ and, of course, the solution $\mathbf{z} \equiv 0$ is stable too.[11]

§ 3

Consider now the equation

$$\frac{d\mathbf{z}}{dt} = \mathbf{A}(t)\mathbf{z} + \mathbf{B}(t)\varphi(\mathbf{z}) \tag{9}$$

where the matrix $\mathbf{A}(t)$ is also a function of t. Concerning this equation we shall prove the following

THEOREM 3. *If*

1. $\mathbf{A}(t)$ *is continuous for* $t \geq 0$ *and* $\varliminf_{t\to\infty} \int_0^t \mathrm{tr}(\mathbf{A})dt > -\infty$ *or, in particular,* $\mathrm{tr}(\mathbf{A}) \equiv 0$ ($\mathrm{tr}(\mathbf{A})$ *means the trace of* $\mathbf{A}(t)$);
2. *every solution of the equation* (2) (*taking* $\mathbf{A}(t)$ *variable*) *is bounded;*
3. $\mathbf{B}(t)$ *is continuous for* $t \geq 0$ *and* $\int_0^\infty \|\mathbf{B}(t)\| dt < \dfrac{\int_0^\infty \frac{dt}{\omega(t)}}{\sup_{t\geq 0} \|\mathbf{Y}\| \cdot \sup_{t \geq 0} \|\mathbf{Y}^{-1}\|}$;[12]
4. $\boldsymbol{\varphi}(\mathbf{z})$ *and* $\omega(u)$ *are as in Theorem* 1;

then every solution of (9) *is bounded, provided that* $\|\mathbf{c}\|$ *is sufficiently small.*

If $\int_0^u \frac{dt}{\omega(t)} = \infty$ $(u > 0)$, *then the solution* $\mathbf{z} \equiv 0$ *is stable for sufficiently small* $\|\mathbf{c}\|$.

PROOF. We make use of the fact that all the solutions of the linear inhomogeneous equation

$$\frac{d\mathbf{u}}{dt} = \mathbf{A}(t)\mathbf{u} + \mathbf{w}(t) \qquad (\mathbf{u}(0) = \mathbf{c}) \tag{10}$$

[11] Compare to BELLMAN's book, p. 36.

[12] As we shall see immediately the boundedness of $\|\mathbf{Y}^{-1}(t)\|$ ought not to be supposed. It is a consequence of the boundedness of $\|\mathbf{Y}(t)\|$ and the condition 1. The function $\mathbf{Y}^{-1}(t)$ satisfies the adjoint equation $\frac{d\mathbf{Z}}{dt} = -\mathbf{Z}\mathbf{A}(t)$.

may be written in the form

$$\mathbf{u}=\mathbf{y}+\int_0^t \mathbf{Y}(t)\mathbf{Y}^{-1}(t_1)\mathbf{w}(t_1)\,dt_1, \tag{11}$$

where $\mathbf{y}$ and $\mathbf{Y}$ are the solutions of the equations

$$\frac{d\mathbf{y}}{dt}=\mathbf{A}(t)\mathbf{y} \qquad (\mathbf{y}(0)=\mathbf{c}) \tag{2'}$$

and

$$\frac{d\mathbf{Y}}{dt}=\mathbf{A}(t)\mathbf{Y} \qquad (\mathbf{Y}(0)=\mathbf{I}), \tag{3'}$$

respectively. As before, we have $\mathbf{y}=\mathbf{Y}\mathbf{c}$.

Applying (11) we get, similarly as in paragraph 1,

$$\mathbf{z}=\mathbf{y}+\int_0^t \mathbf{Y}(t)\mathbf{Y}^{-1}(t_1)\mathbf{B}(t_1)\varphi(\mathbf{z}(t_1))\,dt_1 \tag{12}$$

where $\mathbf{z}(0)=\mathbf{y}(0)=\mathbf{c}$. Hence

$$\|\mathbf{z}\|\leqq\|\mathbf{y}\|+\int_0^t \|\mathbf{Y}(t)\|\,\|\mathbf{Y}^{-1}(t_1)\|\,\|\mathbf{B}(t_1)\|\,\|\varphi(\mathbf{z})\|\,dt_1.$$

The determinant of $\mathbf{Y}(t)$ is given by the formula

$$|\mathbf{Y}|=e^{\int_0^t \mathrm{tr}(\mathbf{A})\,dt}.$$

Therefore, on account of 1, $\|\mathbf{Y}^{-1}(t)\|$ is bounded. Let $c_1=\sup\|\mathbf{Y}\|$, $c_2=\sup\|\mathbf{Y}^{-1}\|$, then $\|\mathbf{y}\|\leqq c_1\|\mathbf{c}\|$ and denoting c_1c_2 by c_3 we get

$$\|\mathbf{z}\|\leqq c_1\|\mathbf{c}\|+c_3\int_0^t \|\mathbf{B}(t_1)\|\,\omega(\|\mathbf{z}\|)\,dt_1 \qquad (t\geqq 0)$$

whence for sufficiently small $\|\mathbf{c}\|$

$$\|\mathbf{z}\|\leqq \Omega^{-1}\Big(\Omega(c_1\|\mathbf{c}\|)+c_3\int_0^t \|\mathbf{B}(t_1)\|\,dt_1\Big) \qquad (t\geqq 0)$$

and the further conclusion is obvious. Concerning the stability of $\mathbf{z}=0$ we can argue as proving Theorem 1.

An application. Let Theorem 3 apply to the *scalar* non-linear second order equation

$$v''+a(t)v=b(t)\varphi(v,v'). \tag{13}$$

Let us consider simultaneously the linear homogeneous equation

$$u''+a(t)u=0 \tag{14}$$

too. Now we can prove the following

THEOREM 4. *If*

1. $a(t)$ *and* $b(t)$ *are continuous for* $t \geqq 0$;

2. *every solution of* (14) *is bounded together with its derivative;*

3. $\varphi(x, y)$ *is continuous in some region about* $x=y=0$ *and* $\omega(u)$ *is like formerly, i. e.* $\omega(u) \geqq \max\limits_{|\zeta_1|+|\zeta_2|\leqq u} |\varphi(\zeta_1, \zeta_2)|$;

4. $\int\limits_0^t |b(t)|dt < \dfrac{\int\limits_0^\infty \dfrac{dt}{\omega(t)}}{\sup\limits_{t\geqq 0} \|\mathbf{Y}\| \cdot \|\sup\limits_{t\geqq 0} \|\mathbf{Y}^{-1}\|}$;[13]

then the same holds for every solution of (13), *provided that* $|v(0)|+|v'(0)|$ *is sufficiently small. If* $\int\limits_0^1 \dfrac{dt}{\omega(t)} = \infty$, *the solution* $v=0$ *is stable, provided that* $|v(0)|+|v'(0)|$ *is sufficiently small.*

PROOF. By the denotations $u=y_1$, $u'=y_2$, $v=z_1$, $v'=z_2$, the equations (13) and (14) will have the forms

$$\frac{d\mathbf{z}}{dt} = \mathbf{A}(t)\mathbf{z} + \mathbf{B}(t)\psi(\mathbf{z}), \tag{13'}$$

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}(t)\mathbf{y}, \tag{14'}$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -a(t) & 0 \end{pmatrix} \quad (\mathrm{tr}\,\mathbf{A} = 0!), \qquad \mathbf{B}(t) = \begin{pmatrix} 0 & 0 \\ b(t) & 0 \end{pmatrix}, \qquad \psi(\mathbf{z}) = \begin{pmatrix} \varphi(z_1, z_2) \\ 0 \end{pmatrix}.$$

If we take into account that $\omega(u) \geqq \max\limits_{|z_1|+|z_2|\leqq u} |\varphi(z_1, z_2)| = \max\limits_{\|\mathbf{z}\|\leqq u} \|\psi(\mathbf{z})\|$, we see that the hypotheses of Theorem 3 are all satisfied relative to (13′) and (14′). Thus the theorem is proved.

§ 4

We regard the equation

$$\frac{d\mathbf{z}}{dt} = \mathbf{A}(t)\mathbf{z} + \mathbf{B}(t)\varphi(\mathbf{z}) \qquad (\mathbf{z}(0) = \mathbf{c}) \tag{15}$$

and simultaneously

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}(t)\mathbf{y} \qquad (\mathbf{y}(0) = \mathbf{c}). \tag{16}$$

[13] Here is $\mathbf{Y} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$, where y_1 and y_2 are the solutions of (14) with $y_1(0)=1$, $y_1'(0)=0$, $y_2(0)=0$, $y_2'(0)=1$.

THEOREM 5. *If*

1. $\mathbf{A}(t)$, $\mathbf{B}(t)$ *are continuous for* $t \geq 0$ *and* $\mathbf{A}(t)$ *is periodic;*

2. $\int_0^\infty \|\mathbf{B}(t)\|\,dt < \dfrac{\int_0^\infty \frac{dt}{\omega(t)}}{k}$ *(*$k>0$ *is a constant;)*

3. $\varphi(\mathbf{z})$ *is as previously;*

4. *every solution of* (16) *is bounded;*

then the same is true for every solution of (15), *provided that* $\|\mathbf{c}\|$ *is sufficiently small and* k *sufficiently large. If* $\int_0^a \frac{dt}{\omega(t)}$ *is divergent,* $\mathbf{z}=0$ *is stable supposed that* $\|\mathbf{c}\|$ *is small enough.*

Assuming condition 4 of Theorem 2, all the solutions of (15) tend to zero as $t\to\infty$, provided that the same is true for the solutions of (16).

In order to *prove* this theorem we employ the known result that the solution of the equation

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}(t)\mathbf{Y} \qquad (\mathbf{Y}(0)=\mathbf{I}) \tag{17}$$

has the form $\mathbf{Y}=\mathbf{P}(t)e^{\mathbf{C}t}$, where $\mathbf{P}(t)$ is a periodic matrix with the period of $\mathbf{A}(t)$ and $\mathbf{C}$ is a constant matrix.

If every solution of (16) is bounded, $e^{\mathbf{C}t}$ must be bounded as $t\to\infty$. If every solution of (16) tends to zero, then $e^{\mathbf{C}t}$ tends exponentially to 0, i. e. $\|e^{\mathbf{C}t}\| \leq c_1 e^{-at}$ $(c_1>0,\ a>0)$.

Since

$$\mathbf{z} = \mathbf{y} + \int_0^t \mathbf{Y}(t)\mathbf{Y}^{-1}(t_1)\mathbf{B}(t_1)\varphi(\mathbf{z}(t_1))\,dt_1 = \mathbf{y} + \int_0^t \mathbf{P}(t)e^{\mathbf{C}(t-t_1)}\mathbf{P}^{-1}(t_1)\mathbf{B}(t_1)\,\varphi(\mathbf{z}(t_1))\,dt_1$$

(where $\mathbf{y}$ is the solution of (16)), we have

$$\|\mathbf{z}\| \leq c_2\|\mathbf{c}\| + k\int_0^t \|\mathbf{B}(t_1)\|\,\omega(\|\mathbf{z}\|)\,dt_1$$

(c_2 means an upper bound for $\|\mathbf{Y}\|$), whence the boundedness follows as before (e. g. in the proof of Theorem 1). The stability of $\mathbf{z}=0$ will be proved similarly as in the previous cases, finally the second part of the theorem may be proved as in Theorem 2.

These results may be applied to the Mathieu and Hill equations.

§ 5

THEOREM 6. *If in the non-linear equation*

$$\frac{d\mathbf{z}}{dt} = (\mathbf{A} + \mathbf{B}(t))\mathbf{z} + \mathbf{C}(t)\boldsymbol{\varphi}(\mathbf{z}) \tag{18}$$

1. $\mathbf{A}$ *is a constant matrix all of whose characteristic roots have non-positive real parts, while those with zero real parts are simple;*

2. $\mathbf{B}(t) \to 0$ *as* $t \to \infty$ *and* $\int^{\infty} \left\| \frac{d\mathbf{B}}{dt} \right\| dt < \infty$;

3. $\int^{\infty} \|\mathbf{C}(t)\| dt < \infty$;

4. *the characteristic roots of* $\mathbf{A} + \mathbf{B}(t)$ *have non-positive real parts for* $t \geqq t_0$;

5. $\|\boldsymbol{\varphi}(\mathbf{T}\mathbf{z})\| \leqq \omega(\|\mathbf{z}\|)$, *where* $\mathbf{T}$ *is a matrix figuring below,* $\omega(u)$ *is as formerly and* $\mathbf{z}$ *is arbitrary;*

6. $\boldsymbol{\varphi}(\mathbf{z})$ *is as above;*

then all the solutions of (18) *are bounded.*

This is a generalization of a theorem of L. CESARI.[14] The proof follows the same lines with slight changes. Great parts of CESARI's proof will be here simply reproduced.

PROOF. Let the simple characteristic roots of $\mathbf{A}$ which possess zero real parts be denoted by $\lambda_1, \lambda_2, \ldots, \lambda_k$, then, as known, there exists a matrix $\mathbf{T}$ such that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} \lambda_1 & & & 0 & d_{1,k+1} & \cdots & d_{1n} \\ & \lambda_2 & & & & & \\ & & \ddots & & & & \\ & & & \lambda_k & d_{k,k+1} & \cdots & d_{kn} \\ & & & & \lambda_{k+1} & & \\ & & & & & \ddots & \\ & & & & & & \lambda_n \end{pmatrix}. \tag{19}$$

The notation signifies that the initial $k \cdot k$ submatrix is diagonal and that all the elements beneath the main diagonal are zero. The elements of $\mathbf{T}$ may be chosen to be polynomials in the elements and characteristic roots of $\mathbf{A}$.

We denote the characteristic roots of $\mathbf{A} + \mathbf{B}(t)$ by $\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)$. Let $\lambda_1(t), \lambda_2(t), \ldots, \lambda_k(t)$ be the characteristic roots of $\mathbf{A} + \mathbf{B}(t)$ which approach

[14] L. CESARI, Sulla stabilità delle soluzioni delle equazioni differenziali lineari, *Annali R. Scuola Norm. Sup. Pisa,* Ser. 2, **8** (1939), pp. 131—148.

$\lambda_1, \lambda_2, \ldots, \lambda_k$ and which, as known, are simple for all sufficiently large t values. The remaining characteristic roots $\lambda_{k+1}(t), \ldots, \lambda_n(t)$ have negative real parts for large t.

Let $\mathbf{T}(t)$ be the matrix corresponding to $\mathbf{T}$ above, formed so that $\mathbf{T}^{-1}(t)(\mathbf{A}+\mathbf{B}(t))\mathbf{T}(t)$ has a form similar to (19) with $\lambda_i(t)\to\lambda_i$. In this way we have $\mathbf{T}(t)\to\mathbf{T}$ as $t\to\infty$ and therefore $\mathbf{T}^{-1}(t)$ is uniformly bounded as $t\to\infty$. Furthermore the assumption $\int^{\infty}\left\|\frac{d\mathbf{B}}{dt}\right\|dt<\infty$ yields the result that $\int^{\infty}\left\|\frac{d\mathbf{T}}{dt}\right\|dt<\infty$.

Substituting $\mathbf{z}=\mathbf{T}(t)\mathbf{w}$ in (18) we obtain

$$\frac{d\mathbf{w}}{dt}=\mathbf{T}^{-1}(\mathbf{A}+\mathbf{B}(t))\mathbf{T}\mathbf{w}-\mathbf{T}^{-1}\frac{d\mathbf{T}}{dt}\mathbf{w}+\mathbf{T}^{-1}\mathbf{C}\boldsymbol{\varphi}(\mathbf{T}\mathbf{w}). \tag{20}$$

By virtue of 3 and the above establishments concerning $\mathbf{T}(t)$ we get for the matrices $-\mathbf{T}^{-1}\frac{d\mathbf{T}}{dt}=\mathbf{R}(t)$, $\mathbf{T}^{-1}\mathbf{C}=\mathbf{S}(t)$

$$\int^{\infty}\|\mathbf{R}\|\,dt<\infty,\qquad \int^{\infty}\|\mathbf{S}\|\,dt<\infty.$$

Writing equation (20) out in terms of its components, we have

$$\frac{dw_i}{dt}=\lambda_i(t)w_i+\sum_{j=k+1}^{n}d_{ij}(t)w_j+\sum_{j=1}^{n}r_{ij}(t)w_j+\sum_{j=1}^{n}s_{ij}(t)\varphi_j(\mathbf{T}\mathbf{w}) \tag{21a}$$
$$(i=1,2,\ldots,k),$$

$$\frac{dw_i}{dt}=\lambda_i(t)w_i+\sum_{j=i+1}^{n}d_{ij}(t)w_j+\sum_{j=1}^{n}r_{ij}(t)w_j+\sum_{j=1}^{n}s_{ij}(t)\varphi_j(\mathbf{T}\mathbf{w}) \tag{21b}$$
$$(i=k+1,\ldots,n),$$

where $d_{ij}(t)\to d_{ij}=$ const as $t\to\infty$ and $\int^{\infty}|r_{ij}|\,dt<\infty$, $\int^{\infty}|s_{ij}|\,dt<\infty$.

The difference between (21a) and (21b) is manifested in the summation over the terms $d_{ij}w_j$, a difference which is a consequence of the form (19).

Let us discuss the solution of (21b). Taking the case $i=n$ first we find

$$w_n=c_n e^{\int_0^t \lambda_n(t_1)dt_1}+\int_0^t e^{\int_{t_1}^t \lambda_n(s)ds}\left(\sum_{j=1}^{n} r_{nj}(t_1)w_j(t_1)\right)dt_1+$$
$$+\int_0^t e^{\int_{t_1}^t \lambda_n(s)ds}\left(\sum_{j=1}^{n} s_{nj}(t_1)\varphi_j(\mathbf{T}\mathbf{w})\right)dt_1.$$

(We have made use of the form of the solution of a linear inhomogeneous equation.)

Since the real parts of the $\lambda_i(t)$ for $i=k+1, \ldots, n$ are negative for large t, we have, for some positive constant a,

$$|w_n| \leq |c_n| e^{-at} + \int_0^t e^{-a(t-t_1)} \|\mathbf{R}\| \, \|\mathbf{w}\| \, dt_1 + \int_0^t e^{-a(t-t_1)} \|\mathbf{S}\| \omega(\|\mathbf{w}\|) dt_1 .$$

Let $\max\limits_{t \geq 0} (\|\mathbf{R}\|, \|\mathbf{S}\|)$ be denoted by $U(t)$, then

$$(22) \qquad |w_n| \leq c_n \, e^{-at} + \int_0^t e^{-a(t-t_1)} U(t_1)(\|\mathbf{w}\| + \omega(\|\mathbf{w}\|)) dt_1 .$$

Take now the case $i=n-1$, then we get

$$w_{n-1} = c_{n-1} e^{\int_0^t \lambda_{n-1}(t_1) dt_1} + \int_0^t e^{\int_{t_1}^t \lambda_{n-1}(s) ds} d_{n-1,n}(t_1) w_n(t_1) dt_1 +$$

$$+ \int_0^t e^{\int_{t_1}^t \lambda_{n-1}(s) ds} \left(\sum_{j=1}^n r_{n-1,j}(t_1) w_j(t_1) \right) dt_1 + \int_0^t e^{\int_{t_1}^t \lambda_{n-1}(s) ds} \left(\sum_{j=1}^n s_{n-1,j}(t_1) \varphi_j(\mathbf{T}\mathbf{w}) \right) dt_1 .$$

Hence

$$|w_{n-1}| \leq |c_{n-1}| e^{-at} + \int_0^t e^{-a(t-t_1)} |d_{n-1,n}(t_1)| \, |w_n(t_1)| \, dt_1 + \int_0^t e^{-a(t-t_1)} \|\mathbf{R}\| \, \|\mathbf{w}\| \, dt_1 +$$

$$+ \int_0^t e^{-a(t-t_1)} \|\mathbf{S}\| \omega(\|\mathbf{w}\|) dt_1 .$$

Making use of (22) we obtain ($d_{n-1,n}(t)$ being bounded: $|d_{n-1,n}(t)| < c_{n+1}$)

$$|w_{n-1}| \leq |c_{n-1}| e^{-at} + \int_0^t e^{-a(t-t_1)} U(t_1)(\|\mathbf{w}\| + \omega(\|\mathbf{w}\|)) dt_1 +$$

$$+ c_{n+1} \int_0^t e^{-a(t-t_1)} \left[|c_n| e^{-at_1} + \int_0^{t_1} e^{-a(t_1-t_2)} U(t_2)(\|\mathbf{w}\| + \omega(\|\mathbf{w}\|)) dt_2 \right] dt_1 .$$

The first term in the second integral yields $c_{n+1}|c_n| t e^{-at}$. The second term is

$$c_{n+1} e^{-at} \int_0^t \left(\int_0^{t_1} e^{at_2} U(t_2) (\|\mathbf{w}\| + \omega(\|\mathbf{w}\|)) \, dt_2 \right) dt_1 .$$

Integrated by parts, this becomes

$$c_{n+1} \int_0^t (t-t_1) e^{-a(t-t_1)} U(t_1)(\|\mathbf{w}\| + \omega(\|\mathbf{w}\|)) dt_1 .$$

Since $te^{-at} < b_1 e^{-a_1 t}$ for $t \ge 0$ if $a_1 < a$ and if b_1 is suitable chosen, we obtain

$$|w_{n-1}| \le b_2 e^{-a_1 t} + b_3 \int_0^t e^{-a_1(t-t_1)} U(t_1)(\|\mathbf{w}\| + \omega(\|\mathbf{w}\|))\,dt_1 .$$

Since $a_1 < a$, we obtain the same inequality for $|w_n|$ by increasing the values of b_2 and b_3 if necessary.

Continuing in this way step by step, we find that constants b_4, b_5 and a_1 exist such that

$$|w_i| \le b_4 e^{-a_1 t} + b_5 \int_0^t e^{-a_1(t-t_1)} U(t_1)(\|\mathbf{w}\| + \omega(\|\mathbf{w}\|))\,dt_1 \qquad (k+1 \le i \le n). \tag{23}$$

Turning to the equation where $1 \le i \le k$ we obtain

$$w_i = c_i e^{\int_0^t \lambda_i(t_1)dt_1} + \int_0^t e^{\int_{t_1}^t \lambda_i(t_2)dt_2} \left(\sum_{j=k+1}^n d_{ij}(t_1) w_j \right) dt_1 +$$
$$+ \int_0^t e^{\int_{t_1}^t \lambda_i(t_2)dt_2} \left(\sum_{j=1}^n r_{ij}(t_1) w_j \right) dt_1 + \int_0^t e^{\int_{t_1}^t \lambda_i(t_2)dt_2} \left(\sum_{j=1}^n s_{ij}(t_1) \varphi_j(\mathbf{Tw}) \right) dt_1 .$$

Since the real parts of $\lambda_1(t), \ldots, \lambda_k(t)$ are non-positive, we have for $i = 1, 2, \ldots, k$

$$|w_i| \le c_i + c_i' \int_0^t \left(\sum_{j=k+1}^n |w_j| \right) dt_1 + \int_0^t \|\mathbf{R}\| \|\mathbf{w}\|\, dt_1 + \int_0^t \|\mathbf{S}\| \omega(\|\mathbf{w}\|)\, dt_1 . \tag{24}$$

From (23)

$$\sum_{j=k+1}^n |w_j| \le c_2' e^{-a_1 t} + c_3' \int_0^t e^{-a_1(t-t_1)} U(t_1)(\|\mathbf{w}\| + \omega(\|\mathbf{w}\|))\,dt_1 .$$

Since

$$\int_0^t \left(\int_0^{t_1} e^{-a_1(t_1-t_2)} F(t_2)\,dt_2 \right) dt_1 = -\frac{1}{a_1} \int_0^t e^{-a_1(t-t_1)} F(t_1)\,dt_1 + \frac{1}{a_1} \int_0^t F(t_1)\,dt_1 ,$$

we get from (24)

$$|w_i| \le c_4' + c_5' \int_0^t U(t_1)(\|\mathbf{w}\| + \omega(\|\mathbf{w}\|))\,dt_1 \qquad (1 \le i \le k).$$

Combining this with (23) we derive

$$\|\mathbf{w}\| \le c_6' + c_7' \int_0^t U(t_1)(\|\mathbf{w}\| + \omega(\|\mathbf{w}\|))\,dt_1 , \tag{25}$$

whence

$$\|\mathbf{w}\| \leqq \Omega^{-1}\Big(\Omega(c_6') + c_7' \int_0^t U(t_1)\, dt_1\Big) \tag{26}$$

where

$$\Omega(u) = \int_{u_0}^{u} \frac{dt}{t + \omega(t)} \qquad (u_0 > 0,\ u \geqq 0).$$

(Viz. $\overline{\omega}(u) = u + \omega(u)$ is an increasing function.) But

$$\int_0^t U(t_1) dt_1 \leqq \int_0^t \|\mathbf{R}\| dt_1 + \int_0^t \|\mathbf{S}\| dt_1 .$$

Thus $\int_0^\infty U dt < \infty$, therefore by virtue of (26) $\mathbf{w}$ is bounded and $\mathbf{z} = \mathbf{T}(t)\mathbf{w}$ too.

REMARK. It is easy to present cases where the condition 5 is satisfied.

§ 6

Theorems 7, 8, 9 involve assertions concerning boundedness and limit of solutions and its derivatives of certain non-linear differential equations of second order. These theorems are generalizations of similar theorems relative to linear second order equations.[15]

THEOREM 7. *If in the equation*[16]

$$u'' + \varphi(t) f(u) h(u') = 0 \tag{27}$$

1. *$h(z)$ is a positive continuous function defined for every real z;*

2. *$f(u)$ is a continuous function and $\int_0^{\pm\infty} f(u)\, du = \pm\infty$, further* $\operatorname{sg} f(u) = \operatorname{sg} u$;

3. *$\varphi(t) \geqq \delta > 0$ for $t \geqq 0$ and $\int_0^\infty |\varphi'(t)| dt < \infty$;*

then every solution of (27) *is bounded for* $t \geqq 0$.

PROOF. From (27) we have

$$\frac{u' u''}{h(u')} + \varphi(t) f(u) u' = 0.$$

[15] See BELLMAN's book, pp. 112—115. At these equations the degree of non-linearity is so great that a discussion by matrices is impossible.

[16] In connection with the same equation (27) and analogous types oscillation and monotonity problems will be discussed in a forthcoming paper of the author.

Introducing the denotations $\int_0^z \frac{s}{h(s)}\,ds = H(z)$, $\int_0^z f(s)ds = F(z)$ $(H(z) \geqq 0$ for all z and $\lim_{z \to \pm\infty} F(z) = \pm\infty)$ we get

(28) $$\frac{dH(u')}{dt} + \varphi(t)\frac{dF(u)}{dt} = 0.$$

Hence

$$H(u') + \varphi(t)F(u) - \int_0^t \varphi'(t)F(u)dt = c_1,$$

further

$$\delta|F(u)| \leqq c_2 + \int_0^t |\varphi'(t)||F(u)|\,dt \qquad (c_2 = |c_1|),$$

whence by our lemma $(\omega(u) \equiv u)$

$$|F(u)| \leqq \frac{c_2}{\delta} e^{\frac{1}{\delta}\int_0^t |\varphi'(t)|\,dt},$$

i. e. $F(u)$ is bounded what is only possible when u is too. Of course, it may be here $h(z) \equiv 1$, $f(u) \equiv u$.

Given $F(u)$ we have an explicit bound for u at least in principle.

THEOREM 8. *If in the equation*

(29) $$u'' + \varphi(t)f(u)h(u') = 0$$

1. $\operatorname{sg} f(u) = \operatorname{sg} u$ *and* $h(z) > 0$ *for arbitrary* z;
2. $\varphi(t) > 0$, *non-decreasing and bounded for* $t \geqq 0$;
3. $H(+\infty) = \infty$ *where* $H(z) = \int_0^z \frac{t}{h(t)}\,dt$;

then $|u'|$ *is bounded for* $t \geqq 0$.

PROOF. From (28) of the previous paragraph

$$\frac{1}{\varphi(t)}\frac{dH(u')}{dt} + \frac{dF(u)}{dt} = 0$$

where $F(u) = \int_0^u f(t)dt$. Hence

$$\frac{H(u')}{\varphi(t)} + \int_0^t \frac{\varphi'(t_1)}{\varphi(t_1)^2} H(u')dt_1 + F(u) = c_1,$$

therefore $\frac{H(u')}{\varphi(t)} \leqq c_1$ whence $H(u') \leqq c_1\varphi(t)$ and our assertion is proved.

E. g. in linear case $\left(f(u)=u,\ h(z)=1,\ H(z)=\frac{z^2}{2}\right)$ $|u'|\leq c_2|\varphi(t)|$.

THEOREM 9. *If in the equation*

$$u''+a(t)\varphi(u)=0 \tag{30}$$

1. $\int^{\infty} t|a(t)|dt<\infty$;

2. $\frac{|\varphi(u)|}{t}\leq\omega\left(\frac{|u|}{t}\right)$ *for all* u *and* $t\geq 1$ $\Big($*where* $\omega(u)$ *is as in the Introduction and* $\int_{1}^{\infty}\frac{dt}{\omega(t)}=+\infty\Big)$;

then $\lim\limits_{t\to+\infty} u'$ *exists.*

PROOF. Two times integrating (29) from 1 to t we get

$$u=c_1+c_2t-\int_1^t(t-t_1)a(t_1)\varphi(u(t_1))dt_1. \tag{31}$$

From this we obtain for $t\geq 1$

$$|u|\leq(|c_1|+|c_2|)t+t\int_1^t|a(t_1)||\varphi(u(t_1))|dt_1$$

or

$$\frac{|u|}{t}\leq|c_1|+|c_2|+\int_1^t t_1|a(t_1)|\frac{|\varphi(u)|}{t_1}dt_1,$$

and applying 2

$$\frac{|u|}{t}\leq|c_1|+|c_2|+\int_1^t t_1|a(t_1)|\omega\left(\frac{|u|}{t_1}\right)dt_1,$$

whence by our lemma and 1, $\frac{|u|}{t}\leq k=\text{const}$ for $t\geq 1$. But

$$u'=c_2-\int_1^t a(t_1)\varphi(u)dt_1$$

and

$$\int_1^t|a(t_1)||\varphi(u)|dt_1\leq\int_1^t t_1|a(t_1)|\omega\left(\frac{|u|}{t_1}\right)dt_1\leq\omega(k)\int_1^t t_1|a(t_1)|dt_1,$$

therefore $\int_1^{\infty}a(t_1)\varphi(u(t_1))dt_1$ exists and $\lim\limits_{t\to+\infty}u'$ too.

(E. g. if $\varphi(u)=\omega(u)=\sqrt[3]{u}$, then 2 is satisfied for $t \geqq 1$.)

We assert that there exists a solution u with the property $\lim_{t\to-\infty} u' \neq 0$. Take t_0 so large that $\omega(k)\int_{t_0}^{t} t_1 |a(t_1)| dt_1 < \frac{1}{2}$ what is possible regarding 1 and consider the solution of (30) with $u'(t_0)=1$, then $u'(t)=1-\int_{t_0}^{t} a(t_1)\varphi(u(t_1))dt_1$. From this

$$|u'| > 1-\int_{t_0}^{t} |a(t_1)||\varphi(u)|\,dt_1 >$$

$$> 1-\int_{t_0}^{t} t_1 |a(t_1)|\,\omega\left(\frac{|u|}{t_1}\right)dt_1 \geqq 1-\omega(k)\int_{t_0}^{t} t_1 |a(t_1)|\,dt_1 > \frac{1}{2} > 0.$$

That is to say, there exists a solution of the form $u \sim ct$ for $t\to+\infty$ where $c \neq 0$. Obviously, the theorem holds when $|\varphi(u)| \leqq |u|$.

An example where $\lim_{u=\pm 0} \frac{\varphi(u)}{u} = \pm\infty$ and 2 is fulfilled, is $\varphi(u) \equiv \omega(|u|) = |u| + \sqrt[3]{|u|}$.

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ON LIMITING DISTRIBUTIONS CONCERNING A SOJOURN TIME PROBLEM

By

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(*Presented by* A. RÉNYI)

Introduction

In an earlier paper [8] the author dealt with the following problem: Consider a stochastic process $\{\xi(t), 0 \leqq t < \infty\}$ whose random variables take on values in X where X is an arbitrary abstract space. Let $X = A + B$ where A and B are fixed disjoint sets. Define a new stochastic process $\{\chi(t), 0 \leqq t < \infty\}$ as follows:

$$\chi(t) = \begin{cases} 1 & \text{if } \xi(t) \in B, \\ 0 & \text{if } \xi(t) \in A. \end{cases} \tag{1}$$

Suppose $\xi(0) \in A$. Then the process assumes the states $A, B, A, B, A, \ldots$ alternatingly. Denote by $\xi_1, \eta_1, \xi_2, \eta_2, \xi_3, \ldots$ the times spent in states A and B, respectively. We suppose that $\xi_1, \eta_1, \xi_2, \eta_2, \xi_3, \ldots$ are non-negative independent random variables with the distribution functions $\mathbf{P}\{\xi_n \leqq x\} = G(x)$ and $\mathbf{P}\{\eta_n \leqq x\} = H(x)$ $(n = 1, 2, 3, \ldots)$. Let us introduce the following random variable:

$$\beta(t) = \int_0^t \chi(u)\, du \tag{2}$$

and put $\mathbf{P}\{\beta(t) \leqq x\} = \Omega(t, x)$.

We have proved in [8] that

$$\Omega(t, x) = \sum_{n=0}^{\infty} H_n(x)\, [G_n(t-x) - G_{n+1}(t-x)] \tag{3}$$

where $H_n(x)$ and $G_n(x)$ denote the n-times iterated convolution of the distribution function $H(x)$ and $G(x)$, respectively, with itself ($H_0(x) = 1$ if $x \geqq 0$, $H_0(x) = 0$ if $x < 0$ and $G_0(x) \equiv 1$). Further we have proved the following

THEOREM 1. *If* $\mathbf{D}\{\xi_n\} = \sigma_\alpha$ *and* $\mathbf{D}\{\eta_n\} = \sigma_\beta$ *exist, then putting* $\mathbf{M}\{\xi_n\} = \alpha$, $\mathbf{M}\{\eta_n\} = \beta$ *we have*

$$\lim_{t \to \infty} \mathbf{P}\left\{ \frac{\beta(t) - \dfrac{\beta t}{\alpha + \beta}}{\sqrt{\dfrac{\beta^2 \sigma_\alpha^2 + \alpha^2 \sigma_\beta^2}{(\alpha+\beta)^3}\, t}} \leqq x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}}\, du. \tag{4}$$

In what follows we shall determine the asymptotic distribution of the random variable $\beta(t)$ in other cases.*

§ 1. The asymptotic distribution of $\beta(t)$

We expect that $\beta(t)$ has an asymptotic distribution if the random variables $\xi_1+\xi_2+\cdots+\xi_n$ and $\eta_1+\eta_2+\cdots+\eta_n$ have asymptotic distributions as $n\to\infty$. According to the central limit theorem these sums have an asymptotic normal distribution if σ_α and σ_β are finite, respectively. If the dispersion is infinite, then for the existence of an asymptotic distribution a necessary and sufficient condition was given by W. DOEBLIN [3] (cf. W. FELLER [4]). According to this theorem, the following conditions are needed:

$$1-G(x)=x^{-\gamma_1}h_1(x)$$

and

$$1-H(x)=x^{-\gamma_2}h_2(x),$$

where γ_1 $(0<\gamma_1<2)$ and γ_2 $(0<\gamma_2<2)$ are constants and

$$\lim_{x\to\infty}\frac{h_i(cx)}{h_i(x)}=1 \qquad (i=1,2)$$

for every positive constant c.

In the following we shall make a restriction supposing that

(5) $$\lim_{x\to\infty}[1-G(x)]\,x^{\gamma_1}=A$$

and

(6) $$\lim_{x\to\infty}[1-H(x)]\,x^{\gamma_2}=B,$$

where $0<\gamma_1<2, 0<\gamma_2<2$, A and B are positive constants. We remark that the expectation α exists if $1<\gamma_1<2$ and $\alpha=\infty$ if $0<\gamma_1\le 1$. For the expectation β similar statements hold.

Finally, we mention that according to the central limit theorem

(7) $$\lim_{n\to\infty}G_n(n\alpha+xn^{\frac{1}{2}}\sigma_\alpha)=\Phi(x)$$

if $\sigma_\alpha<\infty$ and

(8) $$\lim_{n\to\infty}H_n(n\beta+xn^{\frac{1}{2}}\sigma_\beta)=\Phi(x)$$

if $\sigma_\beta<\infty$, uniformly in x $(-\infty<x<\infty)$. $\Phi(x)$ denotes the normal distribution function with mean 0 and dispersion 1. Further if $G(x)$ and $H(x)$ satisfy the conditions (5) and (6), respectively (cf. W. DOEBLIN [3], W. FELLER

* Cf. also L. TAKÁCS, On a sojourn time problem, Теория Вероятностей и ее Применения (under press).

[4]), then

(9) $$\lim_{n\to\infty} G_n(n\alpha + x(nA)^{\frac{1}{\gamma_1}}) = F_{\gamma_1}(x)$$

if $1<\gamma_1<2$, uniformly in x $(-\infty<x<\infty)$, and

(10) $$\lim_{n\to\infty} G_n(x(nA)^{\frac{1}{\gamma_1}}) = F_{\gamma_1}(x)$$

if $0<\gamma_1<1$, uniformly in x $(0<x<\infty)$. Similarly,

(11) $$\lim_{n\to\infty} H_n(n\beta + x(nB)^{\frac{1}{\gamma_2}}) = F_{\gamma_2}(x)$$

if $1<\gamma_2<2$ and

(12) $$\lim_{n\to\infty} H_n(x(nB)^{\frac{1}{\gamma_2}}) = F_{\gamma_2}(x)$$

if $0<\gamma_2<1$.

Here $F_\gamma(x)\,(0<x<\infty)$ denotes the stable distribution function whose characteristic function is given by

(13) $$\varphi_\gamma(z) = \exp\left\{-|z|^\gamma\left(\cos\frac{\pi\gamma}{2} - i\sin\frac{\pi\gamma}{2}\operatorname{sign} z\right)\Gamma(1-\gamma)\right\}$$

if $\gamma \neq 1$.

In the sequel we suppose that $G(x)$ satisfies $\sigma_\alpha<\infty$ or (5) and $H(x)$ satisfies $\sigma_\beta<\infty$ or (6) and we shall investigate the possible limiting distributions. The following scheme shows the domain of each theorem:

$H(x)$ \ $G(x)$	$\sigma_\alpha<\infty$	$1<\gamma_1<2$	$0<\gamma_1<1$
$\sigma_\beta<\infty$	Th. 1	Th. 7	Th. 5
$1<\gamma_1<2$	Th. 6	Th. 7 : $\gamma_1>\gamma_2$ Th. 2 : $\gamma_1=\gamma_2$ Th. 6 : $\gamma_1<\gamma_2$	Th. 5
$0<\gamma_1<1$	Th. 4	Th. 4	Th. 8 : $\gamma_1>\gamma_2$ Th. 3 : $\gamma_1=\gamma_2$ Th. 9 : $\gamma_1<\gamma_2$

THEOREM 2. *Suppose that the distribution functions $G(x)$ and $H(x)$ satisfy* (5) *and* (6), *respectively, with the same $\gamma=\gamma_1=\gamma_2$ where $1<\gamma<2$.*

Then the expectations α and β are finite and we have for every x

$$(14)\qquad \lim_{t\to\infty} \mathbf{P}\left\{ \frac{\beta(t)-\dfrac{\beta t}{\alpha+\beta}}{\left(\dfrac{t}{\alpha+\beta}\right)^{1/\gamma}} < x \right\} = -\int_{-\infty}^{\infty} F_\gamma\left(\frac{\beta y+x}{B^{1/\gamma}}\right) d_y F_\gamma\left(\frac{\alpha y-x}{A^{1/\gamma}}\right).$$

The distribution function on the right-hand side can be expressed as follows:

$$(15)\qquad \mathbf{P}\left\{ \frac{\alpha B^{\frac{1}{\gamma}}\xi-\beta A^{\frac{1}{\gamma}}\eta}{\alpha+\beta} < x \right\}$$

where ξ and η are independent random variables with the same stable distribution function $F_\gamma(x)$.

REMARK 1. It is easy to see that (15) is also a stable distribution function whose characteristic function is

$$\exp\left\{ -\frac{(\beta^\gamma A+\alpha^\gamma B)\,|z|^\gamma}{(\alpha+\beta)^\gamma}\left(\cos\frac{\pi\gamma}{2}+i\,\frac{\beta^\gamma A-\alpha^\gamma B}{\beta^\gamma A+\alpha^\gamma B}\sin\frac{\pi\gamma}{2}\operatorname{sign} z\right)\Gamma(1-\gamma)\right\}.$$

If, particularly, $\beta^\gamma A=\alpha^\gamma B$, then (15) is a symmetric stable distribution function whose characteristic function is

$$\exp\left\{ -\frac{2\beta^\gamma A\cos\frac{\pi\gamma}{2}\,\Gamma(1-\gamma)\,|z|^\gamma}{(\alpha+\beta)^\gamma}\right\}.$$

THEOREM 3. *Suppose that the distribution functions $G(x)$ and $H(x)$ satisfy* (5) *and* (6), *respectively, with the same $\gamma=\gamma_1=\gamma_2$ where $0<\gamma<1$. Then we have for $0<x<1$*

$$(16)\qquad \lim_{t\to\infty}\mathbf{P}\{\beta(t)\le tx\} = -\int_0^\infty F_\gamma\left(\frac{x}{yB^{1/\gamma}}\right) d_y F_\gamma\left(\frac{1-x}{yA^{1/\gamma}}\right).$$

The distribution function on the right-hand side can be expressed as follows:

$$(17)\qquad \mathbf{P}\left\{ \frac{\xi}{\eta} < \left(\frac{A}{B}\right)^{\frac{1}{\gamma}} \frac{x}{1-x} \right\}$$

where ξ and η are independent random variables with the same stable distribution function $F_\gamma(x)$.

An important particular case is the following

COROLLARY. *Suppose that* $\lim_{x\to\infty}[1-G(x)]\,x^{1/2}=A$ *and* $\lim_{x\to\infty}[1-H(x)]\,x^{\frac{1}{2}}=B$

with positive constants A and B, then

$$\lim_{t\to\infty} \mathbf{P}\{\beta(t)\leqq tx\}=\begin{cases} 0 & \textit{if} \quad x\leqq 0,\\ \dfrac{2}{\pi}\arc\sin\sqrt{\dfrac{A^2x}{A^2x+B^2(1-x)}} & \textit{if} \quad 0<x<1,\\ 1 & \textit{if} \quad x\geqq 1.\end{cases} \tag{18}$$

To prove (18) we remark that

$$F_{\frac{1}{2}}(x)=\begin{cases} 2\left[1-\Phi\left(\sqrt{\dfrac{\pi}{2x}}\right)\right] & \text{if} \quad x\geqq 0,\\ 0 & \text{if} \quad x<0.\end{cases}$$

Consequently, the random variables ξ and η in Theorem 2 may be replaced by $\xi=\pi/2\,\xi^{*2}$ and $\eta=\pi/2\,\eta^{*2}$, respectively, where ξ^* and η^* are independent normally distributed random variables with mean 0 and dispersion 1. Hence in case $\gamma=\frac{1}{2}$, we have by Theorem 2

$$\lim_{t\to\infty}\mathbf{P}\{\beta(t)\leqq tx\}=\mathbf{P}\left\{\frac{\eta^{*2}}{\xi^{*2}}\leqq\left(\frac{A}{B}\right)^2\frac{x}{1-x}\right\}=\mathbf{P}\left\{\left|\frac{\eta^*}{\xi^*}\right|\leqq\frac{A}{B}\sqrt{\frac{x}{1-x}}\right\}.$$

It is well known that the random variable η^*/ξ^* has a Cauchy distribution, i. e. $\mathbf{P}\left\{\frac{\eta^*}{\xi^*}\leqq x\right\}=\frac{1}{2}+\frac{1}{\pi}\arc\operatorname{tg}x$. Consequently,

$$\mathbf{P}\left\{\left|\frac{\eta^*}{\xi^*}\right|\leqq\frac{A}{B}\sqrt{\frac{x}{1-x}}\right\}=\frac{2}{\pi}\arc\operatorname{tg}\frac{A}{B}\sqrt{\frac{x}{1-x}}=\frac{2}{\pi}\arc\sin\sqrt{\frac{A^2x}{A^2x+B^2(1-x)}}$$

what proves (18).

REMARK 2. If, particularly, $A=B$ in the Corollary, then

$$\lim_{t\to\infty}\mathbf{P}\{\beta(t)\leqq tx\}=\begin{cases} 0 & \text{if} \quad x\leqq 0,\\ \dfrac{2}{\pi}\arc\sin\sqrt{x} & \text{if} \quad 0<x<1,\\ 1 & \text{if} \quad x\geqq 1.\end{cases} \tag{19}$$

This result contains as a particular case the following theorem proved earlier by P. LÉVY [6] (cf. W. FELLER [5], p. 252): Let $\delta_1, \delta_2, \ldots, \delta_n, \ldots$ be independent random variables with $\mathbf{P}\{\delta_n=1\}=\mathbf{P}\{\delta_n=-1\}=\frac{1}{2}$. Put $\xi(t)=$ $=\delta_0+\delta_1+\cdots+\delta_{[t]}$ where $\delta_0=0$ and $[t]$ denotes the function "entier t". The stochastic process $\xi(t)$ defines a random walk. We speak about the state B if $\xi(t)>0$ and the state A if $\xi(t)\leqq 0$. In this case $\mathbf{P}\{\xi_1=2j\}=\mathbf{P}\{\eta_1=2j\}=$ $=\frac{1}{2j-1}\binom{2j}{j}\frac{1}{2^{2j}}\sim\frac{1}{2\pi^{\frac{1}{2}}j^{\frac{3}{2}}}$ (as $j\to\infty$), i. e. $1-G(x)=1-H(x)\sim\sqrt{\frac{2}{\pi x}}$ (as $x\to\infty$). Thus (19) holds.

THEOREM 4. *Let us suppose that for the distribution function $G(x)$ we have* a) $\sigma_\alpha < \infty$ *or* b) (5) *holds with* $1 < \gamma_1 < 2$ *and* $H(x)$ *satisfies* (6) *with* $0 < \gamma_2 < 1$. *In this case* $\alpha < \infty$ *and* $\beta = \infty$ *and we have*

$$\lim_{t\to\infty} \mathbf{P}\{\beta(t) \leq t - x t^{\gamma_2}\} = F_{\gamma_2}\left(\left(\frac{\alpha}{Bx}\right)^{1/\gamma_2}\right) \tag{20}$$

if $0 < x < \infty$.

By changing the role of the states A and B in the above theorem we obtain

THEOREM 5. *Let us suppose that for the distribution function $H(x)$ we have* a) $\sigma_\beta < \infty$ *or* b) (6) *holds with* $1 < \gamma_2 < 2$ *and* $G(x)$ *satisfies* (5) *with* $0 < \gamma_1 < 1$. *In this case* $\beta < \infty$ *and* $\alpha = \infty$ *and we have*

$$\lim_{t\to\infty} \mathbf{P}\{\beta(t) \leq x t^{\gamma_1}\} = 1 - F_{\gamma_1}\left(\left(\frac{\beta}{Ax}\right)^{1/\gamma_1}\right). \tag{21}$$

REMARK 3. If, particularly, $\gamma_2 = \frac{1}{2}$ in Theorem 4 and $\alpha(t) = t - \beta(t)$, i.e. $\alpha(t)$ denotes the time spent in state A during the time interval $(0, t)$, then

$$\lim_{t\to\infty} \mathbf{P}\{\alpha(t) \leq x t^{\frac{1}{2}}\} = 2\,\Phi\left(\sqrt{\frac{\pi}{2}\,\frac{Bx}{\alpha}}\right) - 1 \tag{22}$$

if $0 \leq x < \infty$.

This result contains as a particular case the following theorem proved by R. L. DOBRUSHIN [2] (cf. K. L. CHUNG and M. KAC [1]): Let us consider the stochastic process $\{\xi(t), 0 \leq t < \infty\}$ defined in Remark 2. Now suppose that the process is in state A if $\xi(t) = 0, 1, \ldots, c-1$ where c is a positive integer and in state B if $\xi(t)$ takes on some other value. In this case $\mathbf{M}\{\xi_n\} = \alpha = c$ and $\mathbf{P}\{\eta_n = 2j\} = \frac{1}{2j-1}\binom{2j}{j}\frac{1}{2^{2j}}$, i. e. $1 - H(x) \sim \sqrt{\frac{2}{\pi x}}$. Now, using (22), we obtain

$$\lim_{t\to\infty} \mathbf{P}\{\alpha(t) \leq t^{\frac{1}{2}} x\} = 2\,\Phi\left(\frac{x}{c}\right) - 1 \tag{23}$$

if $0 \leq x < \infty$.

THEOREM 6. *Suppose that the distribution function $G(x)$ satisfies* a) $\sigma_\alpha < \infty$ *or* b) (5) *with* $1 < \gamma_1 < 2$ *and* $H(x)$ *satisfies* (6) *with* $1 < \gamma_2 < 2$ *where* $\gamma_1 > \gamma_2$. *Then for* $-\infty < x < \infty$ *we have*

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\beta(t) - \dfrac{\beta t}{\alpha+\beta}}{\left(\dfrac{t}{\alpha+\beta}\right)^{1/\gamma_2}} \leq x\right\} = F_{\gamma_2}\left(\frac{(\alpha+\beta)\,x}{\alpha B^{1/\gamma_2}}\right). \tag{24}$$

By changing the role of the states A and B in the above theorem we obtain the following

THEOREM 7. *Suppose that the distribution function* $G(x)$ *satisfies* (5) *with* $1 < \gamma_1 < 2$ *and* $H(x)$ *satisfies* a) $\sigma_\beta < \infty$ *or* b) (6) *with* $1 < \gamma_2 < 2$ *where* $\gamma_1 < \gamma_2$. *Then for* $0 < x < \infty$ *we have*

$$\lim_{t\to\infty} \mathbf{P}\left\{ \frac{\beta(t) - \dfrac{\beta t}{\alpha+\beta}}{\left(\dfrac{t}{\alpha+\beta}\right)^{1/\gamma_1}} \leq x \right\} = 1 - F_{\gamma_1}\left(\frac{-(\alpha+\beta)\,x}{\beta A^{1/\gamma_1}} \right). \tag{25}$$

THEOREM 8. *Suppose that the distribution function* $G(x)$ *satisfies* (5) *with* $0 < \gamma_1 < 1$ *and* $H(x)$ *satisfies* (6) *with* $0 < \gamma_2 < 1$ *where* $\gamma_1 > \gamma_2$. *Then for* $0 < x < \infty$ *we have*

$$\lim_{t\to\infty} \mathbf{P}\{\beta(t) \leq t - x t^{\frac{\gamma_2}{\gamma_1}}\} = -\int_0^\infty F_{\gamma_2}\left(\frac{1}{y^{\gamma_1} B^{1/\gamma_2}} \right) d_y\, F_{\gamma_1}\left(\frac{x}{y^{\gamma_2} A^{1/\gamma_1}} \right). \tag{26}$$

The distribution function on the right-hand side can be expressed as

$$\mathbf{P}\left\{ \frac{\xi^{\gamma_1}}{\eta^{\gamma_2}} \geq \frac{x^{\gamma_1} B}{A} \right\} \tag{27}$$

where ξ *and* η *are independent random variables with the stable distribution functions* $F_{\gamma_1}(x)$ *and* $F_{\gamma_2}(x)$, *respectively.*

We conclude easily from Theorem 8 the following

THEOREM 9. *Suppose that the distribution function* $G(x)$ *satisfies* (5) *with* $0 < \gamma_1 < 1$ *and* $H(x)$ *satisfies* (6) *with* $0 < \gamma_2 < 1$ *where* $\gamma_1 < \gamma_2$. *Then for* $0 < x < \infty$ *we have*

$$\lim_{t\to\infty} \mathbf{P}\{\beta(t) \leq x t^{\frac{\gamma_1}{\gamma_2}}\} = -\int_0^\infty F_{\gamma_2}\left(\frac{x}{y^{\gamma_1} B^{1/\gamma_2}} \right) d_y\, F_{\gamma_1}\left(\frac{1}{y^{\gamma_2} A^{1/\gamma_1}} \right). \tag{28}$$

The distribution function on the right-hand side can be expressed as

$$\mathbf{P}\left\{ \frac{\eta^{\gamma_1}}{\xi^{\gamma_2}} \leq \frac{A x^{\gamma_2}}{B} \right\} \tag{29}$$

where ξ *and* η *are independent random variables with the stable distribution functions* $F_{\gamma_1}(x)$ *and* $F_{\gamma_2}(x)$, *respectively.*

Concerning the asymptotic behaviour of the expectation of the random variable $\beta(t)$ we shall prove the following

THEOREM 10. *Suppose that the distribution function* $G(x)$ *satisfies* (5) *with* $0<\gamma_1<1$ *and* $H(x)$ *satisfies* (6) *with* $0<\gamma_2<1$. *Then*

(30) $$\lim_{t\to\infty}\frac{\mathbf{M}\{\beta(t)\}}{t}=\frac{B}{A+B} \quad \text{if} \quad \gamma_1=\gamma_2,$$

(31) $$\lim_{t\to\infty}\frac{\mathbf{M}\{\beta(t)\}}{t^{1+\gamma_1-\gamma_2}}=\frac{B\Gamma(1-\gamma_2)}{A\Gamma(1-\gamma_1)\Gamma(1+\gamma_1-\gamma_2)} \quad \text{if} \quad \gamma_1<\gamma_2$$

and

(32) $$\lim_{t\to\infty}\frac{\mathbf{M}\{t-\beta(t)\}}{t^{1+\gamma_2-\gamma_1}}=\frac{A\Gamma(1-\gamma_1)}{B\Gamma(1-\gamma_2)\Gamma(1+\gamma_2-\gamma_1)} \quad \text{if} \quad \gamma_1>\gamma_2.$$

THEOREM 11. *Suppose that* $G(x)$ *satisfies* (5) *with* $0<\gamma_1<1$ *and* $H(x)$ *satisfies* $\beta<\infty$, *then*

(33) $$\lim_{t\to\infty}\frac{\mathbf{M}\{\beta(t)\}}{t^{\gamma_1}}=\frac{\beta\sin\pi\gamma_1}{A\pi};$$

and similarly if $G(x)$ *satisfies* $\alpha<\infty$ *and* $H(x)$ *satisfies* (6) *with* $0<\gamma_2<1$, *then*

(34) $$\lim_{t\to\infty}\frac{\mathbf{M}\{t-\beta(t)\}}{t^{\gamma_2}}=\frac{\alpha\sin\pi\gamma_2}{B\pi}.$$

REMARK 4. In case $\alpha+\beta<\infty$, we have proved in [7] that

$$\lim_{t\to\infty}\frac{\mathbf{M}\{\beta(t)\}}{t}=\frac{\beta}{\alpha+\beta}.$$

§ 2. Proof of the theorems

We need the following

LEMMA. *Let*

(35) $$\Omega=\sum_{n=0}^{\infty}H_n(G_n-G_{n+1})$$

and

(36) $$\Omega^*=\sum_{n=0}^{\infty}H_n^*(G_n^*-G_{n+1}^*),$$

where $\{H_n\},\{G_n\},\{H_n^*\},\{G_n^*\}$ *are sequences of non-increasing non-negative numbers for which* $H_0\leqq G_0\leqq 1$, $H_0^*\leqq 1$ *and* $G_0^*\leqq 1$. *Suppose that for all* $\varepsilon>0$ *there are integers* N *and* M *for which*

(37) $$1-G_N^*<\varepsilon,$$

(38) $$G_M^*<\varepsilon,$$

(39) $$|H_n-H_n^*|<\varepsilon \quad \text{if} \quad N-1\leqq n\leqq M+1$$

and

(40) $$|G_n - G_n^*| < \varepsilon \quad \text{if} \quad N-1 \leq n \leq M+1.$$

Then we have

(41) $$|\Omega - \Omega^*| < 10\,\varepsilon$$

and

(42) $$\left|\Omega - \sum_{n=N}^{M} H_n^*(G_n^* - G_{n+1}^*)\right| < 8\varepsilon.$$

The proof is very easy and will be omitted (cf. [7]).

PROOF OF THEOREM 2. Put in the Lemma

$$H_n = H_n\left(\frac{\beta t}{\alpha+\beta} + x\left(\frac{t}{\alpha+\beta}\right)^{\frac{1}{\gamma}}\right),$$

$$G_n = G_n\left(\frac{\alpha t}{\alpha+\beta} - x\left(\frac{t}{\alpha+\beta}\right)^{\frac{1}{\gamma}}\right),$$

$$H_n^* = F_\gamma\left(\frac{\dfrac{\beta t}{\alpha+\beta} + x\left(\dfrac{t}{\alpha+\beta}\right)^{\frac{1}{\gamma}} - n\beta}{\left(\dfrac{tB}{\alpha+\beta}\right)^{1/\gamma}}\right)$$

and

$$G_n^* = F_\gamma\left(\frac{\dfrac{\alpha t}{\alpha+\beta} - x\left(\dfrac{t}{\alpha+\beta}\right)^{\frac{1}{\gamma}} - n\alpha}{\left(\dfrac{tA}{\alpha+\beta}\right)^{1/\gamma}}\right).$$

In this case we obtain $\Omega = \Omega\left(t, \dfrac{\beta t}{\alpha+\beta} + x\left(\dfrac{t}{\alpha+\beta}\right)^{1/\gamma}\right)$. Further put $N = N_t =$ $= \dfrac{t}{\alpha+\beta} - \left(\dfrac{t}{\alpha+\beta}\right)^{1/\gamma} t^\delta$ and $M = M_t = \dfrac{t}{\alpha+\beta} + \left(\dfrac{t}{\alpha+\beta}\right)^{1/\gamma} t^\delta$ where

$$0 < \delta < (\gamma - 1)/\gamma.$$

Now for any $\varepsilon > 0$ the inequalities (37), (38), (39), (40) are satisfied if t is sufficiently large. Namely,

$$1 - G_N^* = 1 - F_\gamma\left(\frac{-x + \alpha t^\delta}{A^{1/\gamma}}\right) \to 0 \quad \text{if} \quad t \to \infty$$

and similarly

$$G_M^* = F_\gamma\left(\frac{-x - \alpha t^\delta}{A^{1/\gamma}}\right) \to 0 \quad \text{if} \quad t \to \infty.$$

Further, by virtue of (9) we have

$$\left| G_n\left(\frac{\alpha t}{\alpha+\beta}-x\left(\frac{t}{\alpha+\beta}\right)^{\frac{1}{\gamma}}\right)-F_\gamma\left(\frac{\frac{\alpha t}{\alpha+\beta}-x\left(\frac{t}{\alpha+\beta}\right)^{\frac{1}{\gamma}}-n\alpha}{(nA)^{1/\gamma}}\right)\right|<\frac{\varepsilon}{2}$$

if n is large enough and clearly

$$\left|F_\gamma\left(\frac{\frac{\alpha t}{\alpha+\beta}-x\left(\frac{t}{\alpha+\beta}\right)^{1/\gamma}-n\alpha}{(nA)^{1/\gamma}}\right)-F_\gamma\left(\frac{\frac{\alpha t}{\alpha+\beta}-x\left(\frac{t}{\alpha+\beta}\right)^{1/\gamma}-n\alpha}{\left(\frac{tA}{\alpha+\beta}\right)^{1/\gamma}}\right)\right|<\frac{\varepsilon}{2}$$

if $N_t-1\leq n\leq M_t+1$ and t is large enough. Consequently, $|G_n-G_n^*|<\varepsilon$ if $N_t-1\leq n\leq M_t+1$ and t is large enough. Similarly, by virtue of (11) $|H_n-H_n^*|<\varepsilon$ if $N_t-1\leq n\leq M_t+1$ and t is large enough. Hence, by (39), we have for any $\varepsilon>0$

$$\left|\Omega\left(t,\frac{\beta t}{\alpha+\beta}+x\left(\frac{t}{\alpha+\beta}\right)^{\frac{1}{\gamma}}\right)-\sum_{n=0}^{\infty}F_\gamma\left(\frac{\frac{\beta t}{\alpha+\beta}+x\left(\frac{t}{\alpha+\beta}\right)^{\frac{1}{\gamma}}-n\beta}{\left(\frac{tB}{\alpha+\beta}\right)^{1/\gamma}}\right)\cdot\right.$$

$$\left.\cdot\left|F_\gamma\left(\frac{\frac{\alpha t}{\alpha+\beta}-x\left(\frac{t}{\alpha+\beta}\right)^{1/\gamma}-n\alpha}{\left(\frac{tA}{\alpha+\beta}\right)^{1/\gamma}}\right)-F_\gamma\left(\frac{\frac{\alpha t}{\alpha+\beta}-x\left(\frac{t}{\alpha+\beta}\right)^{1/\gamma}-(n+1)\alpha}{\left(\frac{tA}{\alpha+\beta}\right)^{1/\gamma}}\right)\right|\right|<10\varepsilon$$

if t is sufficiently large. Replacing

$$n=\frac{t}{\alpha+\beta}-y_n\left(\frac{t}{\alpha+\beta}\right)^{1/\gamma}$$

we obtain

$$\lim_{t\to\infty}\Omega\left(t,\frac{\beta t}{\alpha+\beta}+x\left(\frac{t}{\alpha+\beta}\right)^{1/\gamma}\right)=$$

$$\lim_{t\to\infty}\sum_{n=0}^{\infty}F_\gamma\left(\frac{\beta y_n+x}{B^{1/\gamma}}\right)\left|F_\gamma\left(\frac{\alpha y_n-x}{A^{1/\gamma}}\right)-F_\gamma\left(\frac{\alpha y_{n+1}-x}{A^{1/\gamma}}\right)\right|.$$

Being $y_n-y_{n+1}=\left(\frac{t}{\alpha+\beta}\right)^{-\frac{1}{\gamma}}\to 0$ as $t\to\infty$ and $y_\infty=-\infty$, $y_0\to\infty$ as $t\to\infty$, we obtain easily that

$$\lim_{t\to\infty}\Omega\left(t,\frac{\beta t}{\alpha+\beta}+x\left(\frac{t}{\alpha+\beta}\right)^{\frac{1}{\gamma}}\right)=\int_{-\infty}^{\infty}F_\gamma\left(\frac{\beta y+x}{B^{1/\gamma}}\right)d_yF_\gamma\left(\frac{\alpha y-x}{A^{1/\gamma}}\right). \tag{43}$$

Q. e. d.

PROOF OF THEOREM 3. Put in the Lemma

$$H_n = H_n(tx), \quad G_n = G_n(t(1-x)),$$

$$H_n^* = F_\gamma\left(\frac{tx}{(nB)^{1/\gamma}}\right), \quad G_n^* = F_\gamma\left(\frac{t(1-x)}{(nA)^{1/\gamma}}\right).$$

Then we obtain $\Omega = \Omega(t, tx)$. Further put $N = N_t = t^{\frac{1}{\gamma} - \delta}$ (where $0 < \delta < \gamma^2/(1+\gamma)$) and $M = M_t = \infty$.

Now for any $\varepsilon > 0$ the inequalities (37), (38), (39), (40) are satisfied if t is sufficiently large. Namely,

$$1 - G_N^* = 1 - F_\gamma\left(\frac{(1-x)\,t^{\frac{\delta}{\gamma}}}{A^{1/\gamma}}\right) \to 0 \quad \text{if} \quad t \to \infty$$

and $G_M^* = 0$. By virtue of (11) and (12) we have

$$|G_n - G_n^*| < \varepsilon$$

and

$$|H_n - H_n^*| < \varepsilon$$

if n is large enough, i. e. if $N_t - 1 \leq n$ and t is large enough.

Hence, by (42), we obtain for any $\varepsilon > 0$

$$\left|\Omega(t, tx) - \sum_{n=N_t}^{\infty} F_\gamma\left(\frac{tx}{(nB)^{1/\gamma}}\right)\left[F_\gamma\left(\frac{t(1-x)}{(nA)^{1/\gamma}}\right) - F_\gamma\left(\frac{t(1-x)}{((n+1)A)^{1/\gamma}}\right)\right]\right| < 8\varepsilon$$

if t is sufficiently large. Writing

$$n = (ty_n)^\gamma$$

we have

$$\lim_{t\to\infty} \Omega(t, tx) = \lim_{t\to\infty} \sum_{n=N_t}^{\infty} F_\gamma\left(\frac{x}{y_n B^{1/\gamma}}\right)\left[F_\gamma\left(\frac{(1-x)}{y_n A^{1/\gamma}}\right) - F_\gamma\left(\frac{(1-x)}{y_{n+1} A^{1/\gamma}}\right)\right].$$

As $\frac{1}{y_n} - \frac{1}{y_{n+1}} \sim \frac{t}{\gamma n^{(1+\gamma)/\gamma}}$, it holds $\left(\frac{1}{y_n} - \frac{1}{y_{n+1}}\right) \to 0$ if $n > N_t$ and $t \to \infty$. Further $y_\infty = \infty$ and $\lim_{t\to\infty} y_{N_t} = 0$. Consequently, it is easy to see that

$$\lim_{t\to\infty} \Omega(t, tx) = -\int_0^\infty F_\gamma\left(\frac{x}{yB^{1/\gamma}}\right) d_y F_\gamma\left(\frac{(1-x)}{yA^{1/\gamma}}\right). \tag{44}$$

Q. e. d.

PROOF OF THEOREM 4. Put in our Lemma

$$H_n = H_n(t - xt^{\gamma_2}), \quad G_n = G_n(xt^{\gamma_2}), \quad H_n^* = F_{\gamma_2}\left(\left(\frac{\alpha}{Bx}\right)^{\frac{1}{\gamma_2}}\right)$$

and

$$G_n^* = \Phi\left(\frac{xt^{\gamma_2} - n\alpha}{\sigma_n n^{1/2}}\right) \quad \text{or} \quad G_n^* = F_{\gamma_1}\left(\frac{xt^{\gamma_2} - n\alpha}{(nA)^{1/\gamma_1}}\right)$$

in case a) and b), respectively. Now we obtain $\Omega = \Omega(t, t - xt^{\gamma_2})$. Further put $N = N_t = \frac{x}{\alpha} t^{\gamma_2} - t^{\gamma_2 - \delta}$ and $M = M_t = \frac{x}{\alpha} t^{\gamma_2} + t^{\gamma_2 - \delta}$ where $0 < \delta < \gamma_2\left(1 - \frac{1}{\gamma_1}\right)$.

The inequalities (37), (38), (39), (40) are satisfied if t is sufficiently large. Namely, it is easy to see that $1 - G_N^* \to 0$ and $G_M^* \to 0$ if $t \to \infty$. Further, by virtue of (12) we have

$$\left| H_n(t - xt^{\gamma_2}) - F_{\gamma_2}\left(\frac{t - xt^{\gamma_2}}{(nB)^{1/\gamma_2}}\right) \right| < \frac{\varepsilon}{2}$$

if n is large enough and clearly

$$\left| F_{\gamma_2}\left(\frac{t - xt^{\gamma_2}}{(nB)^{1/\gamma_2}}\right) - F_{\gamma_2}\left(\left(\frac{\alpha}{xB}\right)^{\frac{1}{\gamma_2}}\right) \right| < \frac{\varepsilon}{2}$$

if $N_t - 1 \leq n \leq M_t + 1$ and t is large enough. Consequently, $|H_n - H_n^*| < \varepsilon$ if $N_t - 1 \leq n \leq M_t + 1$ and t is large enough. Similarly, by (8) or (11) we obtain $|G_n - G_n^*| < \varepsilon$ if $N_t - 1 \leq n \leq M_t + 1$ and t is large enough. Being H_n^* constant we obtain by the aid of (41) that

$$\lim_{t \to \infty} \Omega(t, t - xt^{\gamma_2}) = F_{\gamma_2}\left(\left(\frac{\alpha}{xB}\right)^{\frac{1}{\gamma_2}}\right) \tag{45}$$

what was to be proved.

Theorem 5 can be proved similarly as Theorem 4.

PROOF OF THEOREM 6. Put in our Lemma

$$H_n = H_n\left(\frac{\beta t}{\alpha + \beta} + x\left(\frac{t}{\alpha + \beta}\right)^{\frac{1}{\gamma_2}}\right), \quad G_n = G_n\left(\frac{\alpha t}{\alpha + \beta} - x\left(\frac{t}{\alpha + \beta}\right)^{\frac{1}{\gamma_2}}\right),$$

$$H_n^* = F_{\gamma_2}\left(\frac{(\alpha + \beta)x}{\alpha B^{1/\gamma_2}}\right)$$

and

$$G_n^* = \Phi\left(\frac{\frac{\alpha t}{\alpha + \beta} - x\left(\frac{t}{\alpha + \beta}\right)^{\frac{1}{\gamma_2}} - n\alpha}{\sigma_\alpha n^{1/2}}\right) \quad \text{or} \quad G_n^* = F_{\gamma_1}\left(\frac{\frac{\alpha t}{\alpha + \beta} - x\left(\frac{t}{\alpha + \beta}\right)^{\frac{1}{\gamma_2}} - n\alpha}{(nA)^{1/\gamma_1}}\right)$$

in case a) and b), respectively. Now $\Omega = \Omega\left(t, \frac{\beta t}{\alpha + \beta} + x\left(\frac{t}{\alpha + \beta}\right)^{\frac{1}{\gamma_2}}\right)$. Further put

$$N = N_t = \frac{t}{\alpha + \beta} - \frac{x}{\alpha}\left(\frac{t}{\alpha + \beta}\right)^{\frac{1}{\gamma_2}} - t^{\frac{1}{\gamma_1} + \delta}$$

and

$$M = M_t = \frac{t}{\alpha+\beta} - \frac{x}{\alpha}\left(\frac{t}{\alpha+\beta}\right)^{\frac{1}{\gamma_2}} + t^{\frac{1}{\gamma_1}+\delta}$$

where $0 < \delta < \dfrac{\gamma_1 - \gamma_2}{\gamma_1\gamma_2}$ and $\gamma_1 = 2$ in case a).

The inequalities (37), (38), (39), (40) are satisfied if t is sufficiently large. Namely, it can easily be seen that $1 - G_N^* \to 0$ and $G_M^* \to 0$ if $t \to \infty$. Further, by virtue of (11) we have

$$\left| H_n\left(\frac{\beta t}{\alpha+\beta} + x\left(\frac{t}{\alpha+\beta}\right)^{\frac{1}{\gamma_2}}\right) - F_{\gamma_2}\left(\frac{\dfrac{\beta t}{\alpha+\beta} + x\left(\dfrac{t}{\alpha+\beta}\right)^{\frac{1}{\gamma_2}} - n\beta}{(nB)^{1/\gamma_2}}\right)\right| < \frac{\varepsilon}{2}$$

if n is large enough and clearly

$$\left| F_{\gamma_2}\left(\frac{\dfrac{\beta t}{\alpha+\beta} + x\left(\dfrac{t}{\alpha+\beta}\right)^{1/\gamma_2} - n\beta}{(nB)^{1/\gamma_2}}\right) - F_{\gamma_2}\left(\frac{(\alpha+\beta)x}{\alpha B^{1/\gamma_2}}\right)\right| < \frac{\varepsilon}{2}$$

if $N_t - 1 \leq n \leq M_t + 1$ and t is large enough. Consequently, $|H_n - H_n^*| < \varepsilon$ if $N_t - 1 \leq n \leq M_t + 1$ and t is large enough. Similarly, by (10) we obtain $|G_n - G_n^*| < \varepsilon$ if n is large enough, i. e. $|G_n - G_n^*| < \varepsilon$ if $N_t - 1 \leq n \leq M_t + 1$ and t is large enough. Being H_n^* constant we obtain by the aid of (41) that

$$\lim_{t\to\infty} \Omega\left(t, \frac{\beta t}{\alpha+\beta} + x\left(\frac{t}{\alpha+\beta}\right)^{\frac{1}{\gamma_2}}\right) = F_{\gamma_2}\left(\frac{(\alpha+\beta)x}{\alpha B^{1/\gamma_2}}\right).$$

Q. e. d.

Theorem 7 may be proved similarly.

Proof of Theorem 8. Put in the Lemma

$$H_n = H_n\left(t - x t^{\frac{\gamma_2}{\gamma_1}}\right), \quad G_n = G_n\left(x t^{\frac{\gamma_2}{\gamma_1}}\right)$$

and

$$H_n^* = F_{\gamma_2}\left(\frac{t}{(nB)^{1/\gamma_2}}\right), \quad G_n^* = F_{\gamma_1}\left(\frac{x t^{\frac{\gamma_2}{\gamma_1}}}{(nA)^{1/\gamma_1}}\right).$$

Now we obtain $\Omega = \Omega\left(t, t - x t^{\frac{\gamma_1}{\gamma_2}}\right)$. Further put $N = N_t = t^{\gamma_2 \delta}$ where δ is positive and $M = M_t = \infty$.

Now for any $\varepsilon > 0$ the inequalities (37), (38), (39), (40) are satisfied if t is sufficiently large. Namely,

$$1 - G_N^* = 1 - F_{\gamma_1}\left(\frac{x t^{\delta,\gamma_1}}{A^{1/\gamma_1}}\right) \to 0 \quad \text{if} \quad t \to \infty$$

and $G_M^* = 0$. By virtue of (10) we have

$$|G_n - G_n^*| < \varepsilon$$

if n is large enough, i. e. if $N_t - 1 \leq n$ and t is large enough. Further, by virtue of (12) we have

$$\left| H_n\left(t - x t^{\frac{\gamma_2}{\gamma_1}}\right) - F_{\gamma_2}\left(\frac{t - x t^{\frac{\gamma_2}{\gamma_1}}}{(nB)^{1/\gamma_2}}\right) \right| < \frac{\varepsilon}{2}$$

if n is large enough and clearly

$$\left| F_{\gamma_2}\left(\frac{t - x t^{\frac{\gamma_2}{\gamma_1}}}{(nB)^{1/\gamma_2}}\right) - F_{\gamma_2}\left(\frac{t}{(nB)^{1/\gamma_2}}\right) \right| < \frac{\varepsilon}{2}$$

if $n \geq N_t - 1$ and t is large enough. Consequently, by (42) we obtain

$$\left| \Omega\left(t, t - x t^{\frac{\gamma_2}{\gamma_1}}\right) - \sum_{n=N_t}^{\infty} F_{\gamma_2}\left(\frac{t}{(nB)^{1/\gamma_2}}\right) \left[F_{\gamma_1}\left(\frac{x t^{\frac{\gamma_2}{\gamma_1}}}{(nA)^{1/\gamma_1}}\right) - F_{\gamma_1}\left(\frac{x t^{\frac{\gamma_2}{\gamma_1}}}{((n+1)A)^{1/\gamma_1}}\right) \right] \right| < 8\varepsilon$$

if t is sufficiently large. Writing

$$n = t^{\gamma_2} y_n^{\gamma_1 \gamma_2}$$

we obtain

$$\lim_{t \to \infty} \Omega\left(t, t - x t^{\frac{\gamma_2}{\gamma_1}}\right) = \lim_{t \to \infty} \sum_{n=N_t}^{\infty} F_{\gamma_2}\left(\frac{1}{y_n^{\gamma_1} B^{1/\gamma_2}}\right) \left[F_{\gamma_1}\left(\frac{x}{y_n^{\gamma_2} A^{1/\gamma_1}}\right) - F_{\gamma_1}\left(\frac{x}{y_{n+1}^{\gamma_2} A^{1/\gamma_1}}\right) \right].$$

As $\dfrac{1}{y_n^{\gamma_2}} - \dfrac{1}{y_{n+1}^{\gamma_2}} \sim \dfrac{t}{\gamma_1 n^{(1+\gamma_1)/\gamma_1}}$, it holds $\dfrac{1}{y_n^{\gamma_2}} - \dfrac{1}{y_{n+1}^{\gamma_2}} \to 0$ if $n \geq N_t$ and $t \to \infty$. Further $\lim\limits_{t \to \infty} y_{N_t} = 0$ and $y_\infty = \infty$. Consequently, it is easy to see that

$$\text{(46)} \qquad \lim_{t \to \infty} \Omega\left(t, t - x t^{\frac{\gamma_2}{\gamma_1}}\right) = -\int_0^\infty F_{\gamma_2}\left(\frac{1}{y^{\gamma_1} B^{1/\gamma_2}}\right) d_y F_{\gamma_1}\left(\frac{x}{y^{\gamma_2} A^{1/\gamma_1}}\right).$$

Q. e. d.

Theorem 9 may be proved similarly as above or can be deduced immediately from Theorem 8.

Proof of Theorems 9 and 10. Denote by $B(t)$ and $A(t)$ the expectations of $\beta(t)$ and $\alpha(t) = t - \beta(t)$, respectively. Clearly $B(t) + A(t) = t$, $B(t)$ and $A(t)$ are non-decreasing functions of t. According to [7], we have

$$\int_0^\infty e^{-st}\, dB(t) = \frac{1}{s} \frac{\gamma(s)(1 - \psi(s))}{1 - \gamma(s)\psi(s)} \qquad (\text{if } \Re(s) > 0)$$

and consequently

$$\int_0^\infty e^{-st}\, dA(t) = \frac{1}{s}\,\frac{1-\gamma(s)}{1-\gamma(s)\,\psi(s)} \qquad (\text{if } \Re(s) > 0),$$

where $\gamma(s)$ and $\psi(s)$ denote the Laplace—Stieltjes transforms of $G(x)$ and $H(x)$, respectively. Now let us suppose that s is a positive real number. If $G(x)$ satisfies $\alpha < \infty$, then $\gamma(s) = 1-\alpha s + o(s)$, and if $G(x)$ satisfies (5) with $0 < \gamma_1 < 1$, then $\gamma(s) = 1 - A\Gamma(1-\gamma_1)s^{\gamma_1} + o(s^{\gamma_1})$. Similarly, if $H(x)$ satisfies $\beta < \infty$, then $\psi(s) = 1-\beta s + o(s)$, and if $H(x)$ satisfies (6) with $0<\gamma_2<1$, then $\psi(s) = 1 - B\Gamma(1-\gamma_2)s^{\gamma_2} + o(s^{\gamma_2})$ as $s \to 0$. Now, in case $0 < \gamma_1 \leqq \gamma_2 \leqq 1$, we have

$$\int_0^\infty e^{-st}\, dB(t) \sim \frac{B}{(A+B)s} \quad \text{if} \quad \gamma_1 = \gamma_2$$

and

$$\int_0^\infty e^{-st}\, dB(t) \sim \frac{B\Gamma(1-\gamma_2)}{A\Gamma(1-\gamma_1)\, s^{1+\gamma_1-\gamma_2}} \quad \text{if} \quad \gamma_1 < \gamma_2,$$

and in case $\beta < \infty, 0 < \gamma_1 < 1$, we have

$$\int_0^\infty e^{-st}\, dB(t) \sim \frac{\beta}{A\Gamma(1-\gamma_1)\, s^{\gamma_1}}$$

as $s \to 0$. Similarly,

$$\int_0^\infty e^{-st}\, dA(t) \sim \frac{A\Gamma(1-\gamma_1)}{B\Gamma(1-\gamma_2)\, s^{1+\gamma_2-\gamma_1}}$$

if $0 < \gamma_2 < \gamma_1 < 1$ and

$$\int_0^\infty e^{-st}\, dA(t) \sim \frac{\alpha}{B\Gamma(1-\gamma_2)\, s^{\gamma_2}}$$

if $\alpha < \infty$ and $0 < \gamma_2 < 1$.

Here $B(t)$ and $A(t)$ are non-decreasing functions of t and their Laplace—Stieltjes transforms are asymptotically equal to C/s^μ if $s \to 0$ where C is a positive constant and $\mu \geqq 1$. We conclude by a theorem of O. Szász [7] that $B(t)$ and $A(t)$ are asymptotically equal to $Ct^{\mu-1}/\Gamma(\mu)$ if $t \to \infty$. Thus we obtain (30), (31), (33) for $B(t)$ and (32), (34) for $A(t)$, what was to be proved.

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SUITES FAIBLEMENT CONVERGENTES DE TRANSFORMATIONS NORMALES DE L'ESPACE HILBERTIEN

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1. Introduction. Soit $\mathfrak{H}$ un espace hilbertien et soit $\mathfrak{B}=\mathfrak{B}(\mathfrak{H})$ l'ensemble des transformations linéaires bornées de $\mathfrak{H}$. Pour $\mathfrak{E}\subset\mathfrak{B}$ désignons par $\bar{\mathfrak{E}}$ l'*adhérence* de $\mathfrak{E}$ dans la topologie faible de $\mathfrak{B}$, définie par le système des voisinages:

$$\mathfrak{V}(T_0;\ f_1,\dots,f_n;\ g_1,\dots,g_n;\ \varepsilon)=\{T\in\mathfrak{B}:\ |((T-T_0)f_i,g_i)|<\varepsilon\ (i=1,\dots,n)\}.$$

Soit $\tilde{\mathfrak{E}}$ l'*ensemble des limites* des suites faiblement convergentes tirées de $\mathfrak{E}$. On a évidemment $\tilde{\mathfrak{E}}\subset\bar{\mathfrak{E}}$, mais ces deux ensembles ne coïncident en général que si l'espace $\mathfrak{H}$ est à un nombre fini de dimensions (cf. [3], p. 383). Dans ce qui suit, *nous supposerons toujours que $\mathfrak{H}$ est à une infinité de dimensions, c'est-à-dire que*

$$d=\dim\mathfrak{H}$$

est un nombre cardinal infini, dénombrable ou non.

Envisageons en particulier les ensembles

$$\mathfrak{E}_T,\ \mathfrak{E}_U,\ \mathfrak{E}_A,\ \mathfrak{E}_E$$

constitués, selon les cas, de toutes les contractions T de $\mathfrak{H}$ ($\|T\|\leq 1$), de toutes les transformations unitaires U, de toutes les transformations autoadjointes A telles que $O\leq A\leq I$, et de toutes les projections orthogonales E.

On a évidemment $\mathfrak{E}_T\supset\mathfrak{E}_U$, $\mathfrak{E}_T\supset\mathfrak{E}_A\supset\mathfrak{E}_E$. Halmos [2] a démontré les relations assez surprenantes:

$$\mathfrak{E}_T=\bar{\mathfrak{E}}_U,\qquad \mathfrak{E}_A=\bar{\mathfrak{E}}_E. \tag{1}$$

Dans cette Note, nous allons voir qu'on a même

$$\mathfrak{E}_T=\tilde{\mathfrak{E}}_U,\qquad \mathfrak{E}_A=\tilde{\mathfrak{E}}_E. \tag{2}$$

Comme les inclusions

$$\tilde{\mathfrak{E}}_U\subset\mathfrak{E}_T,\qquad \tilde{\mathfrak{E}}_E\subset\mathfrak{E}_A$$

sont manifestes, ce qu'il faut montrer c'est que pour tout $T\in\mathfrak{E}_T$ on peut trouver des $U_k\in\mathfrak{E}_U$ ($k=1,2,\dots$), et que pour tout $A\in\mathfrak{E}_A$ on peut trouver des $E_k\in\mathfrak{E}_E$ ($k=1,2,\dots$), de sorte que

$$U_k\rightharpoonup T,\quad E_k\rightharpoonup A\qquad (k\to\infty), \tag{3}$$

le signe $\rightharpoonup$ désignant la convergence faible.

La première de ces relations n'entraîne pas que le carré de U_k tende vers le carré de T, etc., mais nous verrons qu'on peut choisir les U_k de sorte que toutes les puissances de U_k tendent simultanément vers les puissances correspondantes de T, et cela même d'une manière uniforme par rapport aux exposants. On démontrera une proposition analogue pour les semi-groupes à un paramètre de contractions. La deuxième proposition (3) sera généralisée, à son tour, à une transformation $A(\lambda)$, fonction non-décroissante du paramètre λ.

Convenons de la *définition* suivante. Pour des transformations $S(x)$, $S_k(x)$ $(k=1, 2, \ldots)$ dépendant d'un paramètre abstrait x, on dira que $S_k(x)$ converge faiblement vers $S(x)$, uniformément en x, lorsqu'on peut attacher à tout $\varepsilon > 0$ et tout couple d'éléments $f, g \in \mathfrak{H}$ un indice $k_0 = k_0(\varepsilon; f, g)$ ne dépendant pas de x, tel que pour $k \geqq k_0$ on ait

$$|(S_k(x)f, g) - (S(x)f, g)| < \varepsilon.$$

Précisons les propositions que nous venons d'indiquer:

I. *Pour toute contraction T de $\mathfrak{H}$ on peut trouver une suite $\{U_k\}$ de transformations unitaires de $\mathfrak{H}$, telle qu'on ait*

$$U_k^n \rightharpoonup T^n \qquad (k \to \infty)$$

et cela uniformément en n $(n=1, 2, \ldots)$. De plus les transformations U_k $(k=1, 2, \ldots)$ peuvent être choisies unitairement équivalentes.

II. *Pour tout semi-groupe à un paramètre, faiblement continu, de contractions $T(s)$ $(s \geqq 0)$ de $\mathfrak{H}$, on peut trouver une suite $\{U_k(s)\}$ de semi-groupes à un paramètre, fortement continus, de transformations unitaires de $\mathfrak{H}$, telle que*

$$U_k(s) \rightharpoonup T(s) \qquad (k \to \infty)$$

uniformément en s. De plus les semi-groupes $\{U_k(s)\}$ $(k=1, 2, \ldots)$ peuvent être choisis unitairement équivalents.

III. *Soit $A(\lambda)$ une transformation autoadjointe bornée de $\mathfrak{H}$, fonction non-décroissante et continue de droite du paramètre λ $(-\infty < \lambda < \infty)$, et telle que $A(-\infty) = \lim\limits_{\lambda \to -\infty} A(\lambda) = O$, $A(+\infty) = \lim\limits_{\lambda \to \infty} A(\lambda) = I$. Il existe alors une suite $\{E_k(\lambda)\}$ de familles spectrales de $\mathfrak{H}$, c'est-à-dire des projections jouissant, comme fonctions de λ, des mêmes propriétés que $A(\lambda)$, telles que*

$$E_k(\lambda) \rightharpoonup A(\lambda) \qquad (k \to \infty)$$

uniformément en λ. De plus les $E_k(\lambda)$ $(k=1, 2, \ldots)$ peuvent être choisies unitairement équivalentes.

On en obtient la deuxième des propositions (3) en choisissant en particulier $A(\lambda) = O$ pour $\lambda < c_1$, $A(\lambda) = A$ pour $c_1 \leqq \lambda < c_2$, $A(\lambda) = I$ pour $c_2 \leqq \lambda$.

D'après les propositions (3) il existe en particulier des suites $\{U_k\}$, $\{E_k\}$ telles que

$$U_k \rightharpoonup \frac{1}{2} I, \quad E_k \rightharpoonup \frac{1}{2} I.$$

On a alors

$$U_k^{-1} = U_k^* \rightharpoonup \frac{1}{2} I \neq \left(\frac{1}{2} I\right)^{-1}, \quad E_k^2 = E_k \rightharpoonup \frac{1}{2} I \neq \left(\frac{1}{2} I\right)^2,$$

ce qui montre que ni l'opération de prendre le carré ni l'opération de prendre l'inverse ne sont pas continues par rapport à la convergence faible des suites. Que celles-ci ne sont pas continues dans la topologie faible de $\mathfrak{B}$, a été indiqué déjà par HALMOS [2].

2. Démonstration des propositions I—III. Commençons par la démonstration de I. Soit donc T une contraction de $\mathfrak{H}$. Il existe alors, dans un certain espace $\mathfrak{H}' \supset \mathfrak{H}$, une transformation unitaire U' telle que

(4) $$T^n f = P' U'^n f \qquad (f \in \mathfrak{H};\ n = 0, 1, \ldots);$$

P' désigne la projection (orthogonale) dans $\mathfrak{H}'$ sur le sous-espace $\mathfrak{H}$. De plus, on peut exiger que $\mathfrak{H}'$ soit sous-tendu par les éléments de la forme $U'^n f$ ($f \in \mathfrak{H}; n = 0, \pm 1, \pm 2, \ldots$); dans ce cas la structure $\{\mathfrak{H}', U', \mathfrak{H}\}$ est déterminée à isomorphie près (cf. [4]), et on a évidemment

(5) $$\dim \mathfrak{H}' = \aleph_0 \cdot \dim \mathfrak{H} = \aleph_0 \cdot d = d$$

puisque d est un nombre cardinal infini.

Soit $\mathfrak{Q}$ un sous-espace de $\mathfrak{H}$ tel que

(6) $$\dim (\mathfrak{H} \ominus \mathfrak{Q}) = d,$$

et désignons par Q et Q', selon les cas, la projection sur $\mathfrak{Q}$ dans $\mathfrak{H}$ et dans $\mathfrak{H}'$. Comme $\mathfrak{Q} \subset \mathfrak{H} \subset \mathfrak{H}'$, on a $QP' = Q'$, donc il découle de (4) que

(7) $$QT^n f = Q' U'^n f \qquad (f \in \mathfrak{H};\ n = 0, 1, \ldots).$$

De (5) et (6) il s'ensuit que

$$\dim (\mathfrak{H}' \ominus \mathfrak{Q}) = \dim (\mathfrak{H} \ominus \mathfrak{Q}).$$

Par conséquent on peut appliquer $\mathfrak{H}'$ sur $\mathfrak{H}$ par une transformation isométrique τ de façon que les éléments du sous-espace commun $\mathfrak{Q}$ restent invariants et que $\mathfrak{H}' \ominus \mathfrak{Q}$ soit appliqué sur $\mathfrak{H} \ominus \mathfrak{Q}$. On a alors évidemment

$$\tau Q' \tau^{-1} = Q \qquad (\text{dans } \mathfrak{H}).$$

Posons

(8) $$\tau U' \tau^{-1} = U \qquad (\text{dans } \mathfrak{H}),$$

c'est une transformation unitaire de $\mathfrak{H}$.

De (7) il résulte, pour $g \in \mathfrak{H}$,

$$Q T^n Q g = Q' U'^n Q g = \tau^{-1} Q U^n \tau Q g = Q U^n Q g,$$

parce que les éléments de $\mathfrak{Q}$ sont invariants par rapport à τ et τ^{-1}. Donc on a

(9) $$Q T^n Q = Q U^n Q \qquad (n = 0, 1, \ldots).$$

Cela étant, faisons une décomposition de l'espace $\mathfrak{H}$ en somme vectorielle d'une suite de sous-espaces $\mathfrak{P}_i$ $(i = 1, 2, \ldots)$, orthogonaux deux-à-deux, et chacun au même nombre cardinal d de dimensions que $\mathfrak{H}$ lui-même. Cela est bien possible puisque $\aleph_0 \cdot d = d$. Soit

$$\mathfrak{Q}_k = \mathfrak{P}_1 \oplus \cdots \oplus \mathfrak{P}_k \qquad (k = 1, 2, \ldots),$$

on a alors

$$\dim (\mathfrak{H} \ominus \mathfrak{Q}_k) = \dim (\mathfrak{P}_{k+1} \oplus \mathfrak{P}_{k+2} \oplus \cdots) = \aleph_0 \cdot d = d.$$

Il existe donc, pour tout k, une transformation unitaire U_k de $\mathfrak{H}$, unitairement équivalente à la transformation U' et telle que

(10) $$Q_k T^n Q_k = Q_k U_k^n Q_k \qquad (k = 1, 2, \ldots; \quad n = 0, 1, 2, \ldots);$$

Q_k est la projection (dans $\mathfrak{H}$) sur $\mathfrak{Q}_k$. En posant

$$S_k(n) = T^n - U_k^n$$

(10) s'écrit aussi sous la forme

$$Q_k S_k(n) Q_k = O,$$

d'où il résulte que

$$S_k(n) = [Q_k + (I - Q_k)] S_k(n) [Q_k + (I - Q_k)] =$$
$$= Q_k S_k(n) (I - Q_k) + (I - Q_k) S_k(n) Q_k + (I - Q_k) S_k(n) (I - Q_k)$$

ou, pour deux éléments arbitraires $f, g \in \mathfrak{H}$,

$$(S_k(n) f, g) = (Q_k S_k(n) (I - Q_k) f, g) + (S_k(n) Q_k f, (I - Q_k) g) +$$
$$+ (S_k(n) (I - Q_k) f, (I - Q_k) g).$$

Puisque

$$\| S_k(n) \| \leq \| T^n \| + \| U_k^n \| \leq 2$$

il en dérive que

$$|(S_k(n) f, g)| \leq 2 \{ \|(I - Q_k) f\| \cdot \|g\| + \|f\| \cdot \|(I - Q_k) g\| +$$
$$+ \|(I - Q_k) f\| \cdot \|(I - Q_k) g\| \}.$$

Or on a

$$\|(I - Q_k) f\|^2 = \| \sum_{k+1}^{\infty} P_i f \|^2 = \sum_{k+1}^{\infty} \| P_i f \|^2 \to 0 \quad \text{lorsque} \quad k \to \infty,$$

et de même pour g au lieu de f, d'où il résulte que, pour $\varepsilon > 0$ donné, on a

$$|(S_k(n) f, g)| < \varepsilon$$

dès que k dépasse un rang dépendant de ε, f, g, mais indépendant de n.

Cela achève la démonstration de I.

La démonstration de II est analogue. On fait usage du théorème de l'auteur (cf. [4]) que le semi-groupe $T(s)$ peut être représenté sous la forme

$$T(s)f = P'U'(s)f \qquad (f \in \mathfrak{H},\ s \geqq 0)$$

moyennant un groupe à un paramètre, fortement continu, de transformations unitaires $U'(s)$ d'un espace $\mathfrak{H}' \supset \mathfrak{H}$; on peut exiger que $\mathfrak{H}'$ soit sous-tendu par les éléments de la forme $U'(s)f$ $(f \in \mathfrak{H}, -\infty < s < \infty)$, ou grâce à la continuité de $U'(s)$ en fonction de s, sous-tendu par éléments $U'(s)f$ avec s rationnels. Il résulte alors que $\dim \mathfrak{H}' \leq \aleph_0 \cdot d = d$. Puis on raisonne comme plus haut, avec $U_k(s)$ et $S_k(s) = T(s) - U_k(s)$ au lieu de U_k^n et $S_k(n)$, on fait usage en particulier de l'inégalité évidente $\|S_k(s)\| \leqq 2$.

La démonstration de III est fondée sur un théorème de Neumark (cf. [4]) suivant lequel la famille $\{A(\lambda)\}$ peut être représentée sous la forme

$$A(\lambda)f = P'E'(\lambda)f \qquad (f \in \mathfrak{H}, -\infty < \lambda < \infty)$$

moyennant une famille spectrale $\{E'(\lambda)\}$ de projections dans un certain espace $\mathfrak{H}' \supset \mathfrak{H}$; on peut exiger que $\mathfrak{H}'$ soit sous-tendu par les éléments de la forme $E'(\lambda)f$ $(f \in \mathfrak{H}, -\infty < \lambda < \infty)$, ou, grâce à la continuité de droite de $E'(\lambda)$, par les éléments de même forme mais avec λ rationnels. On aura alors de nouveau $\dim \mathfrak{H}' \leq d$. Pour les transformations $S_k(\lambda) = A(\lambda) - E_k(\lambda)$ qui interviennent au cours de la démonstration on a évidemment $\|S_k(\lambda)\| \leq 2$, et la démonstration s'achève comme plus haut.

3. Transformations sous-normales. Une conséquence immédiate de I est que, pour toute transformation linéaire bornée S de $\mathfrak{H}$, il existe une suite $\{N_k\}$ de transformations linéaires bornées de $\mathfrak{H}$ telle que, pour tout nombre naturel fixé n, on ait

(11) $$N_k^n \rightharpoonup S^n \qquad (k \to \infty):$$

il suffit de poser $N_k = \|S\| U_k$ où $\{U_k\}$ est une suite de transformations unitaires correspondant au sens de la proposition I à la contraction $\|S\|^{-1}S$. De (11) il s'ensuit que

$$N_k^{*n} \rightharpoonup S^{*n} \qquad (k \to \infty),$$

mais en général il ne s'ensuit pas que

(12) $$N_k^{*m} N_k^n \rightharpoonup S^{*m} S^n \qquad (k \to \infty),$$

la convergence faible ne jouissant pas de la propriété multiplicative. Pour que la représentation (12) soit possible, S doit vérifier une condition additionnelle :

IV. *Pour qu'on puisse attacher à la transformation S de l'espace $\mathfrak{H}$ une suite $\{N_k\}$ de transformations normales bornées de $\mathfrak{H}$ telle que (12) subsiste pour tout couple m, n d'entiers non-négatifs, il faut et il suffit que S soit*

sous-normale, c'est-à-dire qu'elle admette, dans un espace convenable $\mathfrak{H}' \supset \mathfrak{H}$, un prolongement normal borné N'.[1] *Dans ce cas, les transformations N_k peuvent être choisies unitairement équivalentes l'une à l'autre.*

Démonstration. Supposons d'abord que S est sous-normale donc qu'elle admet un prolongement normal borné N' dans un espace $\mathfrak{H}' \supset \mathfrak{H}$. On peut exiger que $\mathfrak{H}'$ soit sous-tendu par les éléments de la forme $N'^{*m}N'^{n}f$ avec $f \in \mathfrak{H}$, m et n entiers non-négatifs; en effet, grâce à la normalité de N' l'ensemble de ces éléments est invariant par rapport à N' et N'^{*}, et embrasse $\mathfrak{H}$ (cas $m=n=0$). $\mathfrak{H}'$ est alors de dimension $\aleph_0 \cdot d = d$. On a pour $f, g \in \mathfrak{H}$

$$(S^{*m}S^{n}f, g) = (S^{n}f, S^{m}g) = (N'^{n}f, N'^{m}g) = (N'^{*m}N'^{n}f, g),$$

ce qui montre que

$$S^{*m}S^{n}f = P'N'^{*m}N'^{n}f \qquad (f \in \mathfrak{H};\quad m, n = 0, 1, \ldots),$$

P' désignant toujours la projection dans $\mathfrak{H}'$ sur $\mathfrak{H}$. En partant de cette représentation on raisonne tout comme dans la démonstration de la proposition I. On rencontre au cours de ce raisonnement des transformations

$$S_k(m, n) = S^{*m}S^{n} - N_k^{*m}N_k^{n}$$

de l'espace $\mathfrak{H}$, où N_k est, pour tout k, unitairement équivalente à N', d'où il vient que

$$\|S_k(m, n)\| \leq \|S\|^{m+n} + \|N'\|^{m+n},$$

évaluation indépendante de k, mais dépendante de m, n. Par conséquent on peut achever la démonstration de la relation $S_k(m, n) \to O$ $(k \to \infty)$ tout comme on a démontré la relation $S_k(n) \to O$ $(k \to \infty)$, mais cette fois-ci cette convergence n'est pas en général uniforme en n, m. D'ailleurs, dans le cas $\|S\| \leq 1$, on a aussi $\|N'\| \leq 1$,[2] et l'évaluation $\|S_k(m, n)\| \leq 2$ entraîne alors que la convergence (12) est uniforme en m, n.

Passons à la démonstration de la proposition inverse. Supposons donc qu'il existe une suite $\{N_k\}$ de transformations linéaires bornées de $\mathfrak{H}$ telle que (12) se vérifie pour $m, n = 0, 1, \ldots$. On a alors pour tout système fini $g_0, g_1, \ldots, g_r$ d'éléments de $\mathfrak{H}$:

$$\sum_{i=0}^{r}\sum_{j=0}^{r}(S^{*i}S^{j}g_j, g_i) = \lim_{k\to\infty}\sum_{i=0}^{r}\sum_{j=0}^{r}(N_k^{*i}N_k^{j}g_j, g_i) = \lim_{k\to\infty}\left(\sum_{j=0}^{r}N_k^{*j}g_j, \sum_{i=0}^{r}N_k^{*i}g_i\right) \geq 0,$$

ce qui entraîne, d'après un théorème de Halmos—Bram [1, 2], que la transformation S est sous-normale.

Cela achève la démonstration.

[1] Toute transformation normale bornée S de $\mathfrak{H}$ est évidemment aussi sous-normale, on peut prendre alors $\mathfrak{H}' = \mathfrak{H}$, $N' = S$. Mais il existe des transformations sous-normales qui ne sont pas normales (cf. Halmos [2]).

[2] Cf. [1], Lemma 3.

4. Fonctions de type positif. Pour terminer, remarquons que le "théorème principal" du Mémoire [4] de l'auteur, concernant des transformations $T(\xi)$, fonctions de type positif sur un $*$-semi-groupe Γ, admet des conséquences analogues, du moins dans une certaine condition de "séparabilité" qui assure que la dimension de l'espace $\mathfrak{H}'$ qui intervient ne dépasse pas celle de $\mathfrak{H}$. On suppose, tout comme dans [4], que

$$\sum_{\xi\in\Gamma}\sum_{\eta\in\Gamma}(T(\xi^*\eta)g(\eta), g(\xi)) \geqq 0,$$

$$\sum_{\xi\in\Gamma}\sum_{\eta\in\Gamma}(T(\xi^*\alpha^*\alpha\eta)g(\eta), g(\eta)) \leqq C_\alpha^2 \sum_{\xi\in\Gamma}\sum_{\eta\in\Gamma}(T(\xi^*\eta)g(\eta), g(\xi))$$

pour toute famille d'éléments $g(\xi)\in\mathfrak{H}$, presque tous égaux à 0, et pour tout $\alpha\in\Gamma$, et on suppose aussi que $T(\varepsilon)=I$ pour l'élément unité $\varepsilon\in\Gamma$. On y ajoute la

Condition de séparabilité. Il existe un sous-ensemble dénombrable Γ_0 de Γ, tel qu'à tout $\alpha\in\Gamma$ on peut associer une suite $\{\alpha_n\}$ tirée de Γ_0 pour laquelle

$$\limsup C_{\alpha_n} < \infty \quad \text{et} \quad T(\xi\alpha_n\eta) \rightharpoonup T(\xi\alpha\eta) \qquad (n\to\infty)$$

quels que soient $\xi, \eta\in\Gamma$.

Dans ces conditions, on a le théorème suivant qui dérive du "théorème principal" tout comme notre proposition I dérive de la relation $T^n = P'U'^n$:

V. *Il existe une suite $\{D_k(\xi)\}$ de représentations de Γ, unitairement équivalentes, par des transformation linéaires bornées de l'espace $\mathfrak{H}$, telle que*

a) $D_k(\xi) \rightharpoonup T(\xi)$ $(k\to\infty)$;

b) $\|D_k(\xi)\| \leqq C_\xi$;

c) *l'équation $T(\xi\alpha\eta)+T(\xi\beta\eta)=T(\xi\gamma\eta)$, vérifiée pour α,β,γ fixés et ξ,η arbitraires, entraîne l'équation $D_k(\alpha)+D_k(\beta)=D_k(\gamma)$ pour tout k;*

d) *la convergence $T(\xi\alpha_n\eta) \rightharpoonup T(\xi\alpha\eta)$ $(n\to\infty)$ pour α_n, α fixés et ξ,η arbitraires, entraîne la convergence $D_k(\alpha_n)\rightharpoonup D_k(\alpha)$ $(n\to\infty)$ pour tout k fixé.*

La convergence faible a) *est uniforme en ξ dans tout sous-ensemble de Γ sur lequel C_ξ est fonction bornée de ξ.*

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Littérature

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SUR LA CONVERGENCE ABSOLUE DE CERTAINS DÉVELOPPEMENTS ORTHOGONAUX

Par

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1. Un théorème connu de SIDON [5] affirme la convergence absolue d'une série trigonométrique lacunaire, lorsqu'elle est le développement de Fourier d'une fonction bornée. Un théorème analogue est valable pour les développements suivant des fonctions de Rademacher (KACZMARZ—STEINHAUS [2]). Les démonstrations de ces théorèmes utilisent des propriétés spéciales du système trigonométrique resp. du système de Rademacher; leurs généralisations différentes ([1], [4], [6], [7]) n'expliquent non plus la raison pourquoi possèdent justement ces deux types de développements orthogonaux les dites propriétés de convergence absolue.

Nous allons montrer que ces phénomènes de convergence se laissent étendre à une classe large de développements orthogonaux, classe qui contient, entre autres, les développements de Fourier lacunaires et les développements de Rademacher comme cas spéciaux.

2. La propriété qui nous permet la généralisation mentionnée est celle de l'orthogonalité multiplicative. Pour la concevoir, envisageons dans l'intervalle $[a, b]$ un système $\{\varphi_n(x)\}$ de fonctions; désignons par $\Pi\{\varphi_n(x)\}$ le système de fonctions suivant: soit

$$n = 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_m} \qquad (0 \leqq \nu_1 < \nu_2 < \cdots < \nu_m)$$

la représentation diadique de l'indice $n \geqq 1$, alors la n-ième fonction $\psi_n(x)$ de $\Pi\{\varphi_n(x)\}$ est

$$\psi_n(x) = \varphi_{\nu_1+1}(x)\varphi_{\nu_2+1}(x)\ldots\varphi_{\nu_m+1}(x).$$

Complétons ce système de produits par la fonction $\psi_0(x) \equiv 1$, rangeons la devant tous les autres $\psi_n(x)$ et nous obtenons le système $\Pi\{\varphi_n(x)\}$. Il est évident que $\{\varphi_n(x)\}$ est un sous-système de $\Pi\{\varphi_n(x)\}$, notamment $\{\varphi_n(x)\} = \{\psi_{2^{n-1}}(x)\}$. On peut dire que $\Pi\{\varphi_n(x)\}$ est en même rapport à $\{\varphi_n(x)\}$ comme le système de Walsh au système de Rademacher; mais, naturellement, $\Pi\{\varphi_n(x)\}$ n'est, en général, ni orthogonal ni complet. Pourtant, on peut dire peut-être que $\Pi\{\varphi_n(x)\}$ est le système de Walsh engendré par $\{\varphi_n(x)\}$.

Nous appellerons $\{\varphi_n(x)\}$ un *système multiplicativement orthogonal,* si les fonctions $\psi_n(x)$ de rang $n \geq 1$ sont orhogonales à $\psi_n(x)$, c'est-à-dire si

$$\int_a^b \psi_n(x)\,dx = 0 \qquad (n = 1, 2, \ldots).$$

Le système $\{\varphi_n(x)\}$ est *normé,* si

$$\int_a^b \varphi_n^2(x)\,dx = 1 \qquad (n = 1, 2, \ldots).$$

Il est *borné,* s'il existe une constante K indépendante de n telle que $|\varphi_n(x)| \leq K$ presque partout $(n = 1, 2, \ldots)$. Enfin, nous entendons par le *II-développement* de la fonction intégrable $f(x)$ la série $\sum_{n=0}^{\infty} c_n \psi_n(x)$ avec les coefficients

$$c_n = \int_a^b f(x)\,\psi_n(x)\,dx.$$

On voit aisément que le système multiplicativement orthogonal $\{\varphi_n(x)\}$ est univoquement déterminé dans le système $\Pi\{\varphi_n(x)\}$, si $\varphi_n(x)$ est presque partout $\neq 0$; c'est à dire que la relation

$$\varphi_n(x) = \varphi_{\nu_1}(x)\,\varphi_{\nu_2}(x)\ldots\varphi_{\nu_m}(x) \qquad (\nu_1 < \nu_2 < \cdots < \nu_m)$$

presque partout n'est jamais réalisée. Car si $n \neq \nu_k$, on obtiendrait en cas contraire

$$\int_a^b \varphi_n(x)\varphi_{\nu_1}(x)\,\varphi_{\nu_2}(x)\ldots\varphi_{\nu_m}(x)\,dx = \int_a^b \varphi_n^2(x)\,dx \neq 0,$$

ce qui contredit à l'hypothèse de l'orthogonalité multiplicative; or si $n = \nu_k$, alors on aurait

$$\varphi_{\nu_1}(x)\ldots\varphi_{\nu_{k-1}}(x)\,\varphi_{\nu_{k+1}}(x)\ldots\varphi_{\nu_m}(x) = 1$$

presque partout, donc

$$\int_a^b \varphi_{\nu_1}(x)\ldots\varphi_{\nu_{k-1}}(x)\varphi_{\nu_{k+1}}(x)\ldots\varphi_{\nu_m}(x)\,dx \neq 0$$

qui est de nouveau une contradiction à l'orthogonalité multiplicative.

3. Nous allons maintenant démontrer notre théorème principal:

Soit $\{\varphi_n(x)\}$ un système multiplicativement orthogonal, normé et borné. Si la série

(1) $$\sum_{n=1}^{\infty} c_n \varphi_n(x)$$

est le H-développement d'une fonction intégrable $f(x)$ unilatéralement bornée, alors

$$\sum_{n=1}^{\infty} c_n < \infty.$$

Remarquons d'abord qu'on peut supposer, sans restreindre la généralité,

(2) $$|\varphi_n(x)| \leq 1 \qquad (n=1, 2, \ldots).$$

Car, en prenant $\varphi'_n(x) = \frac{\varphi_n(x)}{K}$ au lieu de $\varphi_n(x)$ et $c'_n = \frac{c_n}{K}$ au lieu de c_n, la série $\Sigma c'_n \varphi'_n(x)$ est le H-développement de $f(x)$ suivant le système $\{\varphi'_n(x)\}$; or les séries $\Sigma c_n \varphi_n(x)$ et $\Sigma c'_n \varphi'_n(x)$ sont évidemment à la fois absolument convergentes ou divergentes. Nous supposerons donc, en ce qui suit, l'inégalité (2) satisfaite.

Rangeons les termes de la série (1) dans un ordre arbitraire:

(3) $$\sum_{n=1}^{\infty} c_{\nu_n} \varphi_{\nu_n}(x),$$

et désignons par $\psi_n(x)$ la n-ième fonction du système $H\{\varphi_{\nu_n}(x)\}$. En posant

$$s(n, x) = \sum_{k=1}^{n} c_{\nu_k} \varphi_{\nu_k}(x),$$

on obtient tout de suite

$$s(n, x) = \int_a^b f(t) \sum_{k=0}^{2^n-1} \psi_k(t)\, \psi_k(x)\, dt,$$

parce qu'on a par hypothèse

(4) $$\int_a^b f(t)\, \psi_k(t)\, dt = 0$$

pour tout $\psi_k(t) \neq \varphi_{\nu_l}(t)$, puisque nous avons supposé que $\Sigma c_n \varphi_n(x)$ est le H-développement de $f(x)$. Or on tire de la définition des fonctions $\psi_k(x)$ immédiatement que

$$\sum_{k=0}^{2^n-1} \psi_k(t)\, \psi_k(x) = \prod_{k=1}^{n} [1 + \varphi_{\nu_k}(t)\, \varphi_{\nu_k}(x)].$$

Il s'ensuit donc en vertu de (2):

$$\sum_{k=0}^{2^n-1} \psi_k(t)\, \psi_k(x) \geqq 0.$$

Mais la fonction $f(t)$ est unilatéralement bornée, p. ex. $f(t) \leq M$ où M est un nombre positif; on en obtient donc que

$$s(n, x) \leqq M \int_a^b \sum_{k=0}^{2^n-1} \psi_k(t)\, \psi_k(x)\, dt$$

d'où il s'ensuit par l'orthogonalité multiplicative de $\{\varphi_{\nu_n}(x)\}$ que

$$s(n, x) \leqq M \int_a^b \psi_0(t)\, \psi_0(x)\, dt = M(b-a).$$

De plus, en tenant compte de (4), on voit d'une manière analogue que

$$-s(n, x) = -\int_a^b f(t) \sum_{k=1}^{2^n-1} \psi_k(t)\, \psi_k(x)\, dt =$$

$$= \int_a^b f(t) \prod_{k=1}^{n} [1-\varphi_{\nu_k}(t)\, \varphi_{\nu_k}(x)]\, dt \leqq M(b-a).$$

Les sommes $|s(n, x)|$, c'est à dire: les valeures absolues des sommes partielles de la série (1) avec des termes rangés en un ordre arbitraire ne surpassent pas le nombre fixe $M(b-a)$, ce qui entraîne la convergence absolue de la série (1) en tout point x de $[a, b]$; car en cas contraire on pourrait ranger, grâce à un théorème classique de RIEMANN, les termes de la série (1) en un ordre $\Sigma c_{\nu_n} \varphi_{\nu_n}(x)$, tel que $s(n, x)$ atteigne un nombre arbitrairement grand donné d'avance. La série (1) est donc partout absolument convergente, ce qui entraîne d'après un théorème connu (v. [3], p. 154), la convergence absolue de la série des coefficients: c'était justement notre proposition.

4. Il est évident que le théorème de KACZMARZ—STEINHAUS est un corollaire immédiat du nôtre; mais il est moins évident que le théorème de SIDON en est aussi un corollaire. C'est parce que le système trigonométrique peut être considéré comme un système de Walsh engendré par un système trigonométrique lacunaire. Représentons, en effet, le système trigonométrique en forme complexe: $\{e^{inx}\}$ où n prend les valeurs $0, 1, 2, \ldots$ et $-1, -2, \ldots$; et envisageons le système $\{\exp(\pm 3^m i x)\}$ avec $m = 0, 1, \ldots$. On voit aisément que chaque entier n peut être représenté d'une et une seule manière dans la forme

$$n = 3^{\nu_n} \pm 3^{\nu_{n-1}} \pm \cdots \pm 3^{\nu_1} \qquad (\nu_n > \nu_{n-1} > \cdots > \nu_1).$$

En choisissant donc convenablement les signes $\pm$, on a

$$e^{inx} = \prod_{k=1}^{n} \exp(\pm 3^{\nu_k} i x),$$

c'est à dire que

$$\{e^{inx}\} \equiv \Pi \{\exp(\pm 3^n i x)\}.$$

Cette remarque permet d'obtenir le théorème de SIDON comme corollaire de notre théorème:

Si la série trigonométrique

$$\sum_{n=-\infty}^{\infty} c_n e^{i\nu_n x} \qquad \left(\frac{\nu_{n+1}}{\nu_n} \geqq q > 1\right) \tag{5}$$

est le développement de Fourier d'une fonction $f(x)$ intégrable et unilatéralement bornée, alors

(6) $$\sum_{n=-\infty}^{+\infty} |c_n| < \infty.$$

On peut décomposer, en effet, la série (5) en un nombre fini de séries trigonométriques:

$$\sum_{n=-\infty}^{+\infty} c_n e^{i\nu_n x} = \sum_{k=1}^{p} \sum_{m=-\infty}^{+\infty} c_m^{(k)} \exp(i\nu_m^{(k)} x)$$

avec $\nu_{m+1}^{(k)}/\nu_m^{(k)} \geqq 3$ et $\nu_m^{(k)} \neq \nu_n^{(l)}$ lorsque $m \neq n$ ou $k \neq l$. Alors, les différentes sommes finies $\nu_1^{(k)} + \nu_2^{(k)} + \dots + \nu_m^{(k)}$ représentent des entiers différentes (v. p. ex. [8], p. 139), il s'ensuit donc

$$\Pi\{\exp(i\,\nu_m^{(k)} x)\} \subset \{e^{inx}\}$$

et que $\left\{\dfrac{\exp(i\,\nu_m^{(k)} x)}{\sqrt{2\pi}}\right\}$ est un système multiplicativement orthogonal, normé et borné; les prémisses de notre théorème sont donc satisfaites, par suite

$$\sum_{m=-\infty}^{+\infty} |c_m^{(k)}| < \infty$$

pour chaque k, ce qui éqivaut à (6).

Il est à remarquer que, d'après notre théorème, l'hypothèse de SIDON que (5) soit la série de Fourier de $f(x)$ est surabondante; il suffit déjà l'hypothèse que les coefficients c_n avec un indice représentable en forme $n = \sum \pm \nu_m$ disparaissent. Cette remarque peut avoir un certain intérêt lorsque (5) est une série "très lacunaire". Si p. ex. $\nu_n = 2^{2^n}$, la différence entre cette hypothèse et la prémisse originelle de SIDON est considérable.

5. Il est aisé de construire des systèmes $\{\varphi_n(x)\}$ bien différents du système trigonométrique et du système de Rademacher, et tels que notre théorème soit applicable aux développements suivant leurs fonctions. Mentionnons, à titre d'exemple, les systèmes lacunaires de polynômes orthogonaux. Soit $\varrho(x) \geqq 0$ une fonction presque partout positive et intégrable dans l'intervalle $[a, b]$. Il est connu que la "fonction de poids" $\varrho(x)$ détermine un système unique $\{p_n(x)\}$ de polynômes orthogonaux et normés,[1] tels que $p_n(x)$ soit

[1] Bien entendu, les polynômes $p_n(x)$ sont orthogonaux et normés relativement $\varrho(x)$, c'est à dire que

$$\int_a^b \varrho(x)\, p_k(x)\, p_l(x)\, dx = \begin{cases} 0 & \text{pour } k \neq l, \\ 1 & \text{pour } k = l. \end{cases}$$

Dans ce cas, l'intégrabilité d'une fonction désigne l'intégrabilité relativement $\varrho(x)$, ce qui ne change rien dans nos considérations.

exactement de degré n. Nous prétendons que le système lacunaire $\{p_{r_n}(x)\}$ est multiplicativement orthogonal, si $r_{n+1} : r_n \geq q > 2$. Car dans ce cas, en prenant un produit

$$p_{r_{n_1}}(x)\, p_{r_{n_2}}(x) \ldots p_{r_{n_m}}(x) \qquad (r_{n_1} < r_{n_2} < \cdots < r_{n_m}),$$

le produit des premiers $m-1$ facteurs est un polynôme de degré

$$r_{n_1} + r_{n_2} + \cdots + r_{n_{m-1}} \leqq r_{n_{m-1}} \left(1 + \frac{1}{q} + \cdots + \frac{1}{q^{m-2}}\right) < r_{n_{m-1}} \frac{q}{q-1} \leqq r_{n_m},$$

par conséquent $p_{r_{n_m}}(x)$ est orthogonal à ce produit, c'est à dire que

$$\int_a^b \varrho(x)\, p_{r_{n_1}}(x)\, p_{r_{n_2}}(x) \ldots p_{r_{n_m}}(x)\, dx = 0.$$

Ainsi, notre théorème donne lieu au corollaire suivant:

Soit $\dfrac{r_{n+1}}{r_n} \geq 2$ *et* $\{p_{r_n}(x)\}$ *un système borné de polynômes orthogonaux; alors, si*

$$\sum_{n=1}^{\infty} c_n\, p_{r_n}(x)$$

est le II-développement d'une fonction intégrable unilatéralement bornée, il s'ensuit que

$$\sum_{n=1}^{\infty} |c_n| < \infty.$$

En tous cas, il faut remarquer que ce corollaire n'est pas tellement intéressant comme le théorème de Sidon, parce que $\{p_n(x)\}$ n'est pas un système de Walsh engendré par un système lacunaire $\{p_{r_n}(x)\}$. La convergence de la série $\sum c_n$ n'est donc pas assurée par l'hypothèse que les coefficients de $f(x)$ suivant les polynômes $p_n(x)$ disparaissent à l'exception des coefficients c_n.

6. Nous allons maintenant construire une classe de systèmes de fonctions que l'on peut considérer comme des généralisations naturelles du système de Rademacher. Soit à cette fin $F(x)$ une fonction mesurable et bornée dans l'intervalle $\left[0, \frac{1}{2}\right]$ qui n'est soumise qu'à la seule condition

(7) $$\int_0^{\frac{1}{2}} F^2(x)\, dx = \frac{1}{2}.$$

Définissons $\Phi_1(x)$ par la relation $\Phi_1(x)=F(x)$ pour $0\leq x\leq\frac{1}{2}$ et

$$\Phi_1\left(\frac{1}{2}+t\right)=-\Phi_1\left(\frac{1}{2}-t\right) \quad \text{pour} \quad 0<t\leq\frac{1}{2}.$$

La fonction $\Phi_1(x)$ est alors définie en $[0, 1]$ et nous l'étendons sur toute la droite en posant

$$\Phi_1(x+1)=\Phi_1(x).$$

Définissons maintenant, pour $n=2, 3, \ldots$, les fonctions $\Phi_n(x)$ par induction:

$$\Phi_n(x)=\Phi_1(2^{n-1}x).$$

(Les fonctions de Rademacher correspondent au cas spécial $F(x)=1$.)

On voit tout de suite que le système $\{\Phi_n(x)\}$ est multiplicativement orthogonal. En effet, $r_1<r_2<\cdots<r_n$ étant des indices arbitraires, il s'ensuit par définition que le produit $\Phi_{r_2}(x)\,\Phi_{r_3}(x)\ldots\Phi_{r_n}(x)$ atteint toutes ses valeurs dans l'intervalle $[0, 1/2^{r_2-1}]$ et il se répète périodiquement à l'extérieur. Comme $r_1\leq r_2-1$, ce produit a au moins deux périodes dans l'intervalle $[0, 1/2^{r_1-1}]$; il est donc symétrique au centre $\xi=1/2^{r_1}$ de cet intervalle, tandis que $\Phi_{r_1}(x)$ est, par définition, anti-symétrique à ξ, c'est à dire que

$$\Phi_{r_1}(\xi+t)=-\Phi_{r_1}(\xi-t) \qquad \left(0<t\leq\frac{1}{2^{r_1}}\right).$$

Ainsi, le produit $\Phi_{r_1}(x)\,\Phi_{r_2}(x)\ldots\Phi_{r_n}(x)$ est aussi anti-symétrique à ξ, par conséquent

$$\int_0^{1/2^{r_1-1}}\Phi_{r_1}(x)\,\Phi_{r_2}(x)\ldots\Phi_{r_n}(x)\,dx=0$$

d'où il s'ensuit immédiatement

$$\int_0^1\Phi_{r_1}(x)\,\Phi_{r_2}(x)\ldots\Phi_{r_n}(x)\,dx=0.$$

Le système $\{\Phi_n(x)\}$ est donc multiplicativement orthogonal, borné et, en vertu de (7), normé. Il en résulte, d'après notre théorème, une généralisation considérable du théorème de KACZMARZ—STEINHAUS:

Si la série $\sum_{n=1}^{\infty}c_n\,\Phi_n(x)$ est le Π-développement d'une fonction intégrable unilatéralement bornée, alors

$$\sum_{n=1}^{\infty}|c_n|<\infty.$$

(Reçu le 14 mai 1957.)

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ÜBER EINE VERMUTUNG VON G. HAJÓS

Von

J. MOLNÁR (Budapest)

(*Vorgelegt von* G. HAJÓS)

Wir erinnern an folgenden Satz von HELLY:[1]

Es sei in der euklidischen Ebene eine Menge von einfach zusammenhängenden abgeschlossenen und beschränkten Bereichen vorgegeben. Ist der Durchschnitt von je zwei Bereichen zusammenhängend, und besitzen je drei Bereiche einen gemeinsamen Punkt, so besitzen sämtliche Bereiche einen gemeinschaftlichen Punkt.

Auf der Kugelfläche ist dieser Satz nur unter einer gewissen weiteren Bedingung richtig.[2] Es läßt sich die Frage stellen, ob es außer der Kugelfläche und der damit homöomorphen Flächen noch weitere Flächentypen gibt, auf denen der Hellysche Satz in seiner ursprünglichen Gestalt nicht gilt.[3] In diesem Aufsatz beweisen wir folgende Vermutung von G. HAJÓS:

Es sei auf einer mit einer Kugelfläche nicht homöomorphen Fläche eine Menge von Elementarflächenstücken. Ist der Durchschnitt von je zwei Elementarflächenstücken zusammenhängend und besitzen je drei Elementarflächenstücke einen gemeinsamen Punkt, so besitzen sämtliche Elementarflächenstücke einen gemeinschaftlichen Punkt.

Im folgenden bedeute F eine mit einer Kugelfläche nicht homöomorphe Fläche.

Der Beweis des obigen Satzes beruht auf folgendem

HILFSSATZ. *Es sei auf F eine endliche Anzahl von Elementarflächenstücken so vorgegeben, daß der Durchschnitt von je zwei Elementarflächenstücken eine aus wenigstens zwei Punkten bestehende zusammenhängende Punktmenge ist, und der Durchschnitt von je drei Elementarflächenstücken nicht leer ist, so ist die Vereinigungsmenge der Elementarflächenstücke selbst ein Elementarflächenstück.*

[1] E. HELLY, Über Systeme von abgeschlossenen Mengen mit gemeinschaftlichen Punkten, *Monatshefte f. Math. u. Phys.*, **37** (1930), S. 281–302. S. auch J. MOLNÁR, A két dimenziós topológikus Helly-tételről, *Mat. Lapok*, **8** (1957), S. 108–114.

[2] J. MOLNÁR, Über eine Verallgemeinerung auf die Kugelfläche eines topologischen Satzes von Helly, *Acta Math. Acad. Sci. Hung.*, **7** (1956), S. 107–108.

[3] Wir brauchen den Begriff der Fläche im üblichen Sinn. S. z. B. H. SEIFERT und W. THRELFALL, *Lehrbuch der Topologie* (Leipzig, 1934), S. 142.

Der Beweis geht durch Induktion. Wir bezeichnen die Elementarflächenstücke mit $E_1, E_2, \ldots, E_n$ und beweisen den Hilfssatz zuerst für $n = 2$. Wir bezeichnen den Rand von E_i mit H_i. Wir können uns auf den Fall beschränken, daß $H_2 \not\subseteq E_1$. Ist nämlich $H_2 \subset E_1$, so ist entweder $E_1 \supseteq E_2$, in welchem Fall der Hilfssatz trivial ist, oder gilt $E_1 \mathrel{=\!\supset} E_2$, in welchem Fall aber $E_1 \dotplus E_2$ eine mit einer Kugelfläche homöomorphe Fläche ist.

Im Fall $H_2 \not\subseteq E_1$ besteht der Durchschnitt $H_2 \cdot E_1$ aus gewissen Teilbogen $t_1, t_2, \ldots$, die sich auch zu Punkten entarten können. Wir können annehmen, daß nur Bogen vorkommen, und diese außer ihrer beiden Endpunkten keine weiteren gemeinsamen Punkte mit H_1 aufweisen. Im entgegengesetzten Fall ließe nämlich H_2 im Inneren von E_1 so von H_1 "abwickeln", daß $E_1 \cdot E_2$ weiterhin zusammenhängend ist (Fig. 1a, b). Durch eine derartige "Abwick-

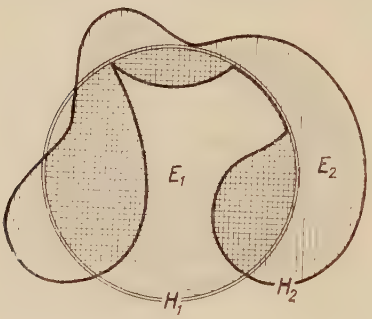

Fig. 1a

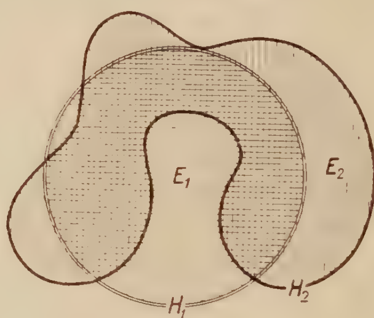

Fig. 1b

lung" bleibt $E_1 \dotplus E_2$ offensichtlich ungeändert. Wir behaupten, daß die zyklische Reihenfolge der Teilbogenendpunkte auf H_2 auch auf H_1 eine zyklische Folge bildet. Das folgt daraus, daß es unter den Teilbogen $t_1, t_2, \ldots$ keinen solchen geben kann, der zwei andere in E_1 voneinander trennt. Wären nämlich t_1 und t_2 voneinander innerhalb E_1 durch t_3 getrennt, so gäbe es einen t_1 und t_2 innerhalb $E_1 \cdot E_2$ verbindenden Bogen b, da ja $E_1 \cdot E_2$ zusammenhängend ist und die Bogen t_1, t_2 enthält (Fig. 2). Der so erhaltene Bogen b muß aber den trennenden Bogen t_3 überqueren, was unmöglich ist, da b im Inneren von E_2 läuft und t_3 am Rande von E_2 liegt.

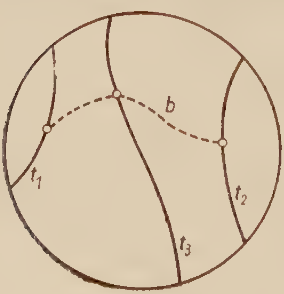

Fig. 2

Also liegen die Bogen $t_1, t_2, \ldots$ bezüglich H_1 zyklisch und begrenzen mit den entsprechenden Bogen von H_1 je ein Elementarflächenstück, die alle fremd zueinander sind. Folglich ergibt sich der Rand der Vereinigung $E_1 + E_2$, indem man aus H_2 die Bogen $t_1, t_2, \ldots$ entfernt, und sie durch je einen,

dieselben Endpunkte verbindenden Teilbogen von H_1 ersetzt. Dies bedeutet aber eben, daß auch die Vereinigung $E_1 + E_2$ ein Elementarflächenstück ist, womit unser Hilfssatz für $n = 2$ bewiesen ist.

Setzen wir nun voraus, daß $\sum_{k=1}^{n-1} E_k$ ein Elementarflächenstück ist. Wir wollen dasselbe für $\sum_{k=1}^{n} E_k$ zeigen. Mit Rücksicht auf die Gültigkeit für $n = 2$ genügt es zu zeigen, daß der Durchschnitt $D = E_n \sum_{k=1}^{n-1} E_k$ eine aus wenigstens zwei Punkten bestehende zusammenhängende Punktmenge ist.

Da nach unseren Voraussetzungen $E_n \cdot E_i$ $(i = 1, \dots, n-1)$ aus mehreren Punkten besteht, gilt dasselbe für D. Wir haben daher nur zu zeigen, daß D zusammenhängend ist. Greifen wir von D zwei beliebige Punkte A und B heraus. Wir können voraussetzen, daß A und B zu verschiedenen Elementarflächenstücken, E_i bzw. E_j gehören. Dann gilt natürlich auch $A \in E_i \cdot E_n$ und $B \in E_j \cdot E_n$. Aus unseren Voraussetzungen folgt einerseits, daß die Bereiche $E_i \cdot E_n$ und $E_j \cdot E_n$ zusammenhängend sind, und anderseits, daß E_i, E_j, E_n einen gemeinsamen Punkt P besitzen. Dieser Punkt P läßt sich also in D sowohl mit A wie mit B verbinden. Folglich ist D zusammenhängend. Damit ist unser Hilfssatz bewiesen.

Es ist beachtenswert, daß jede Bedingung unseres Hilfssatzes nötig ist. Die Vereinigung ist kein Elementarflächenstück, falls 1) der Durchschnitt aus einem einzigen Punkt besteht (Fig. 3), 2) der Durchschnitt nicht zusammenhängend ist (Fig. 4).

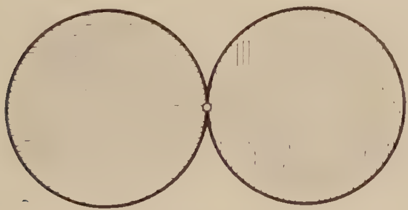

Fig. 3

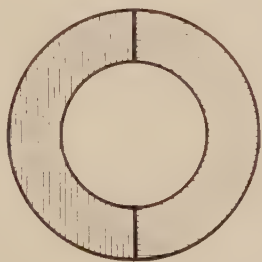

Fig. 4

Wenden wir uns dem Beweise unseres Satzes zu.

Auf Grund eines bekannten Satzes von F. Riesz[1] können wir uns auf

[1] Der Satz von Riesz lautet: Gibt es in einem kompakten Raume eine Menge von geschlossenen Mengen derart, daß der Durchschnitt von je endlich vielen nicht leer ist, so ist der Durchschnitt aller dieser Mengen nicht leer. (S. z. B. B. Kerékjártó, *Vorlesungen über Topologie. I* (Berlin, 1923), S. 37.)

den Fall beschränken, daß nur endlich viele Elementarflächenstücke vorhanden sind.

Setzen wir zunächst voraus, daß der Durchschnitt von je zwei Elementarflächenstücken wenigstens zwei Punkte enthält. Dann ist nach unserem Hilfssatze die Vereinigung der Elementarflächenstücke selbst ein Elementarflächenstück, wodurch die Gültigkeit des Hellyschen Satzes (in seiner ursprünglichen Gestalt) gesichert ist.

Besteht etwa $E_1 \cdot E_2$ aus einem einzigen Punkt A, dann gilt — wegen der Voraussetzung, daß je drei Elementarflächenstücke einen gemeinsamen Punkt aufweisen — $A \in E_1 \cdot E_2 \cdot E_i$, d. h. $A \in E_i$. Da ist die Gültigkeit unseres Satzes auch in diesem Fall bewiesen.

(Eingegangen am 11. Juli 1957.)

ÜBER EINE ÜBERTRAGUNG DES HELLYSCHEN SATZES IN SPHÄRISCHE RÄUME

Von

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(*Vorgelegt von* G. Hajós)

In einer vorigen Arbeit [7][1] habe ich bewiesen den folgenden

Satz A. Ist auf der Kugelfläche eine beliebige Anzahl von einfach zusammenhängenden Bereichen[2] so vorgegeben, daß der Durchschnitt von je zweien zusammenhängend ist, der Durchschnitt von je dreien nicht leer ist, und je vier die Kugel nicht ganz bedecken, so ist der Durchschnitt sämtlicher Bereiche nicht leer.

Im vorliegenden Aufsatz schließen wir uns zu Untersuchungen von H. W. E. Jung [5], C. V. Robinson [8], [9] und Gy. Soós [11].

Im ersten Teil geben wir eine Anwendung des Satzes A auf volle konvexe Flächen, während wir im zweiten Teil unsere Aufmerksamkeit auf mehrdimensionale sphärische Räume richten.

1. Eine zusammenhängende Komponente der Berandung eines konvexen Körpers wird volle konvexe Fläche genannt. Es läßt sich zeigen, daß eine solche Fläche entweder mit der Ebene oder mit der Kugel oder mit dem Zylinder homöomorph ist.[3]

Eine abgeschlossene beschränkte Punktmenge einer vollen konvexen Fläche wird konvexer Bereich genannt, wenn zwei beliebige Punkte der Menge sich eindeutig durch eine kürzeste Flächenkurve verbinden lassen und diese Kurve zur Menge gehört.[4]

[1] Eckige Klammern verweisen auf das Literaturverzeichnis am Ende der Arbeit.

[2] Unter einem Bereiche verstehen wir im folgenden stets eine beschränkte abgeschlossene Punktmenge.

[3] A. D. Alexandrow [1], S. 475.

[4] Diese Erklärung eines konvexen Bereiches ist nicht mit derjenigen von A. D. Alexandrow ([1], S. 94) identisch, die auch mehrfach zusammenhängende Bereiche erlaubt (z. B. auf Zylinderflächen einen Zylinderring). Man kann leicht zeigen, daß mit der Alexandrowschen Erklärung der Hellysche Satz nicht einmal unter der Nebenbedingung gilt, daß je vier Bereiche die volle konvexe Fläche nicht überdecken. Im folgenden Gegenbeispiel besteht die volle konvexe Fläche aus einer von einer Seite mit Halbkugel abgeschlossenen Zylinderfläche. Die „konvexen" Bereiche seien drei kongruente, die Halbkugel überdeckende sphärische Dreiecke und ein zu diesen anschließender Zylinderring. Diese vier Bereiche haben keinen gemeinsamen Punkt.

Da der Durchschnitt von zwei konvexen Bereichen ebenfalls ein konvexer Bereich ist, und die Ebene bzw. der Zylinder mit der einmal bzw. zweimal punktierten Kugelfläche homöomorph ist,[5] haben wir mit Rücksicht auf Satz A den folgenden

Satz I. *Ist auf der vollen konvexen Fläche eine beliebige Anzahl von konvexen Bereichen so vorgegeben, daß der Durchschnitt von je dreien nicht leer ist, und je vier die volle konvexe Fläche nicht ganz bedecken, so ist der Durchschnitt sämtlicher Bereiche nicht leer.*

Dieser Satz schließt gewisse Untersuchungen von Gy. Soós [11] ab.

2. Wir beweisen zunächst den

Satz II. *Ist im n-dimensionalen sphärischen Raum eine beliebige Anzahl von konvexen Körpern so vorgegeben, daß der Durchschnitt von je $n+1$ nicht leer ist, und je $n+2$ den sphärischen Raum nicht bedecken, so ist der Durchschnitt sämtlicher Körper nicht leer.*

Der Beweis ist analog zum Beweis des Satzes A. Mit Rücksicht auf einen Satz von F. Riesz[6] genügt es ihn nur für eine endliche Anzahl von Bereichen zu beweisen. Dies geschieht durch Induktion. Für $n+2$ Bereiche folgt der Satz unmittelbar aus der Tatsache, daß der einmal punktierte n-dimensionale sphärische Raum mit dem n-dimensionalen euklidischen Raum homöomorph ist,[7] so daß die Behauptung sich aus dem bekannten Hellyschen Satz[8] ergibt. Wir setzen nun die Gültigkeit für $k \geq n+2$ Bereiche voraus. Wir betrachten den Durchschnitt $\bar{B}_k = B_k \cdot B_{k+1}$ und behaupten, daß $B_1, B_2, \ldots$ $\ldots, B_{k-1}, \bar{B}_k$ den Bedingungen unseren Satzes Genüge leisten. Es folgt nämlich aus dem schon bewiesenen Fall $k = n+2$, daß $B_{\lambda_1} \cdot B_{\lambda_2} \cdots B_{\lambda_n} \cdot \bar{B}_k = B_{\lambda_1} \cdot B_{\lambda_2} \cdots$ $\cdots B_{\lambda_n} \cdot B_k \cdot B_{k+1}$ $(\lambda_i = 1, 2, \ldots, k-1)$ nicht leer ist. Wegen der Voraussetzung können auch $B_{\lambda_1}, B_{\lambda_2}, \ldots, B_{\lambda_{n+1}}, \bar{B}_k$ den n-dimensionalen sphärischen Raum nicht bedecken, da ja $\bar{B}_k \subset B_k$ gilt. Folglich ist der Satz für die k konvexen Bereiche $B_1, B_2, \ldots, B_{k-1}, \bar{B}_k$ anwendbar, d. h. sie haben einen gemeinsamen Punkt. Dasselbe gilt dann auch für die $k+1$ Bereiche $B_1, B_2, \ldots, B_k, B_{k+1}$. Damit ist der Beweis beendet.

Aus Satz II ergibt sich leicht folgender Satz von Robinson:[9]

[5] S. z. B. Seifert—Threlfall [10], S. 27—28.

[6] Der Satz von Riesz lautet: Gibt es in einem kompakten Raum eine Menge von geschlossenen Mengen derart, daß der Durchschnitt von je endlich vielen nicht leer ist, so ist der Durchschnitt aller dieser Mengen nicht leer. (S. z. B. Kerékjártó [6], S. 37.)

[7] S. z. B. Seifert—Threlfall [10], S. 27—28.

[8] Helly [4]

[9] Robinson [9]. Siehe auch Blumenthal [2], S. 203.

Lassen sich je $n+2$ Punkte einer Punktmenge des n-dimensionalen sphärischen Raumes durch eine Kugel vom Radius $\varrho < \frac{\pi}{2}$ überdecken, so läßt sich die ganze Menge durch eine Kugel vom Radius ϱ bedecken.[10]

BEWEIS. Schlagen wir um jeden Punkt der Menge eine Kugel vom Radius ϱ, so besitzen offenbar je $n+2$ einen gemeinsamen Punkt. Diese $n+2$ Kugeln können aber den Raum nicht bedecken, da der zu dem gemeinsamen Punkt der Kugeln antipodische Punkt — mit Rücksicht auf $\varrho < \frac{\pi}{2}$ — zu keinem dieser Kugeln gehört. Dann besitzen aber nach Satz II sämtliche Kugeln einen gemeinschaftlichen Punkt und die um diesen Punkt mit dem Radius ϱ geschlagene Kugel enthält sämtliche Punkte der Menge.

Als eine zweite Anwendung des Satzes II erwähnen wir folgenden

SATZ III.[11] *Lassen sich je $n+1$ Punkte einer Punktmenge des n-dimensionalen sphärischen Raumes durch eine Kugel vom Radius $\varrho <$ arc cos $\frac{1}{n+1}$*[12] *überdecken, so läßt sich die ganze Menge durch eine Kugel vom Radius ϱ bedecken.*

BEWEIS. Es genügt zu zeigen, daß um je $n+2$ Punkte der Menge geschlagene Kugeln vom Radius ϱ den Raum nicht überdecken. Dies ist aber eine unmittelbare Folgerung eines Satzes von L. FEJES TÓTH,[13] nach dem der Radius von $n+2$ kongruenten Kugeln, die den n-dimensionalen sphärischen Raum überdecken, $\geqq$ arc cos $\frac{1}{n+1}$ ausfällt.

Aus der zweiten Anwendung ergibt sich leicht folgendes sphärisches Analogon des wohlbekannten Jungschen Satzes:[14]

SATZ IV. *Ein im n-dimensionalen sphärischen Raum liegendes Punktsystem von Durchmesser $D <$ arc cos $\frac{-1}{n+1}$*[15] *läßt sich stets durch eine solche*

[10] Betrachten wir in unserem n-dimensionalen sphärischen Raum die $n+2$ Ecken eines regulären $n+1$-dimensionalen Euklidischen Simplexes, so zeigt dieses Punktsystem, daß in diesem Satz die Forderung der Überdeckung von je $n+1$ Punkten nicht genügt.

[11] Für $n=2$ vgl. ROBINSON [9], S. 266–268.

[12] arc cos $\frac{1}{n+1}$ bedeutet den Radius der Umkugel von $n+1$ Punkten des in [10] betrachteten Punktsystems.

[13] FEJES TÓTH [3].

[14] JUNG [5].

[15] arc cos $\frac{-1}{n+1}$ bedeutet den Abstand von zwei Punkten des in [10] betrachteten Punktsystems.

Kugel überdecken, die dem regelmäßigen $(n+1)$-Simplex mit der Kantenlänge D umschrieben ist.

BEWEIS. Mit Rücksicht auf unser voriges Resultat genügt es zu zeigen, daß je $n+1$ Punkte unseren Punktsystems sich durch die Umkugel des regulären Simplexes S der Kantenlänge D überdecken lassen. Um dies einzusehen, betrachten wir den durch die $n+1$ Punkte aufgespannten euklidischen Raum. Nach dem Jungschen Satz lassen sich diese $n+1$ Punkte durch die euklidische Umkugel von S überdecken. Dieser Raum schneidet aus dem sphärischen Raum eine (sphärische) Kugel aus, die die $n+1$ Punkte enthält und die offensichtlich nicht größer ist als die Umkugel von S. Damit ist unser Beweis beendet.

(Eingegangen am 11. Juli 1957.)

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ON A PROBABILITY PROBLEM CONCERNING TELEPHONE TRAFFIC

By

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(*Presented by* A. RÉNYI)

1. Summary. Let us suppose that at a telephone center calls are arriving in the instants $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ $(0 < \tau_1 < \tau_2 < \cdots < \tau_n < \cdots < \infty)$ where $\tau_{n+1} - \tau_n$ $(n = 1, 2, 3, \ldots)$ are identically distributed independent positive random variables with distribution function $F(x)$. There are m available channels. Suppose that a connection is realised if the incoming call finds an idle channel. If all channels are busy, then the incoming call is lost. It is supposed that the durations of the connections are identically distributed independent random variables which are independent also of the process $\{\tau_n\}$ and have the following distribution function:

$$H(x) = \begin{cases} 1 - e^{-\mu x} & \text{if} \quad x \geqq 0, \\ 0 & \text{if} \quad x < 0. \end{cases}$$

Denote by $\eta(t)$ the number of busy channels at the instant t. Define $\eta(\tau_n - 0) = \eta_n$. The sequence of random variables η_n $(n = 1, 2, \ldots)$ forms a Markov chain. There exists a limiting distribution $\lim\limits_{n\to\infty} \mathbf{P}\{\eta_n = j\} = P_j$ $(j = 0, 1, \ldots, m)$ and it is independent of the initial distribution of η_1. Suppose that $\mathbf{P}\{\eta_1 = j\} = P_j$ $(j = 0, 1, \ldots, m)$, then $\{\eta_n\}$ is a stationary Markov chain; let us denote this by $\{\eta_n^*\}$. In case of the stationary Markov chain $\{\eta_n^*\}$ denote by Π_m the probability of a call being lost. (Clearly $\Pi_m = P_m$.)

C. PALM [9] proved that

$$\Pi_m = \frac{1}{\sum\limits_{j=0}^{m} \binom{m}{j} \prod\limits_{k=1}^{j} \frac{1 - \varphi_k}{\varphi_k}} \qquad (m = 1, 2, 3, \ldots) \tag{1}$$

where the empty product means 1 and

$$\varphi_k = \int_0^\infty e^{-k\mu x}\, dF(x). \tag{2}$$

The aim of the present paper is to give a simple proof of (1) and to show that the above problem is identical with a servicing problem.

2. Introduction. In an earlier paper [13] the author determined explicitly the distribution $\{P_j\}$ $(j = 0, 1, \ldots, m)$. But unfortunately it escaped my

attention that C. PALM [9] and F. POLLACZEK [11] have dealt also with this problem. I am indebted to Mr. R. SYSKI for calling my attention that C. PALM and F. POLLACZEK have determined the probability Π_m earlier.* At present we shall determine Π_m in such a way which is simpler than that given by C. PALM and F. POLLACZEK. Actually, the present proof is simpler than that given in [13], although there the aim was to determine the full distribution $\{P_j\}$. The present proof is based on a method given by H. ASHCROFT [1].

3. The probability of a call being lost. Let us consider the stationary Markov chain $\{\eta_n^*\}$ $(n=1, 2, 3, \ldots)$. Following PALM let us suppose that the channels are numbered by $1, 2, \ldots, m$ and an incoming call realises a connection through that idle channel which has the lowest serial number. This assumption does not restrict the generality since the value of Π_m is independent of the system of the handling of traffic. Define Π_r $(r=1, 2, \ldots, m)$ as the probability that an incoming call finds the 1st, 2nd, ..., rth channel to be busy. Denote by Γ_r $(r=1, 2, \ldots, m)$ the expectation of the random number which shows that after a call finding the 1st, 2nd, ..., rth channel to be busy, which is the first call which finds the 1st, 2nd, ..., rth channel also to be busy. Consequently, Γ_m is the expectation of the random number which shows that after a loss-call which is the number of the next loss-call. It follows easily from the theory of Markov chains that $\Gamma_r = 1/\Pi_r$ $(r=1, 2, \ldots, m)$ and particularly

$$\Pi_m = 1/\Gamma_m. \tag{3}$$

Thus the problem is reduced to the determination of Γ_m. We can write easily for $r=1, 2, \ldots, m$ that

$$\Gamma_r = 1 + q_{r,1}\Gamma_r + q_{r,2}(\Gamma_{r-1}+\Gamma_r) + \cdots + q_{r,r}(\Gamma_1+\Gamma_2+\cdots+\Gamma_r) \tag{4}$$

where

$$q_{r,j} = \binom{r}{j}\int_0^\infty e^{-(r-j)\mu x}(1-e^{-\mu x})^j\,dF(x) \qquad (j=0, 1, \ldots, r). \tag{5}$$

Namely, let us consider a call which finds the 1st, 2nd, ..., rth channel to be busy. In order to get Γ_r the next call always must be taken into account If during the time interval between the two mentioned calls exactly j connections $(j=1, 2, \ldots, r)$ come to an end among the connections taking place in the 1st, 2nd, ..., rth channel (the probability of which is $q_{r,j}$), then under this condition the expectation of the number of the further calls until the first call which finds the 1st, 2nd, ..., rth channel to be busy is $\Gamma_{r-j+1}+\cdots+\Gamma_r$;

* Further we remark that J. W. COHEN [2] has obtained nearly simultaneously the same results as the author [13] and so COHEN's paper contains also the explicit form of Π_m.

the role of the channels in the group $(1, 2, \ldots, r)$ being interchangeable. We obtain (4) by the theorem of total expectation.

REMARK 1. C. PALM [9] considered also the expectations Γ_r $(r = 1, 2, \ldots, m)$ but he determined these by stating a system of integral equations for the distribution functions of the inter-arrival times between the calls in question. Further we remark that the expectations Γ_r may be defined for all positive integers r. Namely, the group consisting of the 1st, 2nd, ..., rth channel forms a full group of r channels and the operation of this group is independent of the operations of the other channels. Thus it can also be allowed that the number of the channels is infinite. We investigate, however, only the group consisting of the 1st, 2nd, ..., mth channel.

The unknown Γ_r $(r = 1, 2, \ldots, m)$ can be determined easily from (4) by the aid of the calculus of finite differences. (Cf. CH. JORDAN [4].) Following H. ASHCROFT [1] we proceed as follows:

Put $\Gamma_0 = 1$. Define $T_0 = 0$ and $T_{r+1} = \Gamma_0 + \Gamma_1 + \cdots + \Gamma_r$ $(r = 0, 1, \ldots, m)$. Express the Γ's in (4) by T's and take into consideration that

$$\sum_{k=0}^{r} q_{r,k} = 1,$$

then we obtain

$$T_r = \sum_{k=0}^{r} q_{r,r-k} T_{k+1} - 1. \tag{6}$$

Define the difference operation Δ as usual. Then we can write

$$\Gamma_r = \sum_{j=0}^{r} \binom{r}{j} \Delta^j \Gamma_0 \tag{7}$$

where $\Delta^0 \Gamma_0 = \Gamma_0 = 1$. Thus it is sufficient to determine the unknowns $\Delta^j \Gamma_0$ $(j = 1, 2, \ldots, m)$. Since $\Gamma_0 = \Delta T_0$, we have $\Delta^j \Gamma_0 = \Delta^{j+1} T_0$. As it is well known we can write

$$\Delta^j T_0 = \sum_{r=0}^{j} (-1)^{j-r} \binom{j}{r} T_r. \tag{8}$$

Substituting (6) in (8) and using the identity

$$\sum_{r=k}^{j} (-1)^{j-r} \binom{j}{r} q_{r,r-k} = (-1)^{j-k} \binom{j}{k} q_j, \tag{9}$$

we obtain for $j \geqq 1$

$$\Delta^j T_0 = \sum_{r=0}^{j} (-1)^{j-r} \binom{j}{r} \sum_{k=0}^{r} q_{r,r-k} T_{k+1} = \sum_{k=0}^{j} T_{k+1} \sum_{r=k}^{j} (-1)^{j-r} \binom{j}{r} q_{r,r-k} =$$

$$q_j = \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} T_{k+1} = q_j (\Delta^{j-1} T_0 + \Delta^j T_0).$$

Hence

(10) $$\Delta^{j+1} T_0 = \frac{1-\varphi_j}{\varphi_j} \Delta^j T_0 \qquad (j=1,2,\ldots,m).$$

By applying this formula repeatedly and using $\Delta T_0 = \Gamma_0 = 1$ we obtain

(11) $$\Delta^j \Gamma_0 = \Delta^{j+1} T_0 = \frac{1-\varphi_1}{\varphi_1} \cdot \frac{1-\varphi_2}{\varphi_2} \cdots \frac{1-\varphi_j}{\varphi_j} \qquad (j=1,2,\ldots,m),$$

and according to (7)

(12) $$\Gamma_r = \sum_{j=0}^{r} \binom{r}{j} \prod_{k=1}^{j} \frac{1-\varphi_k}{\varphi_k} \qquad (r=1,2,\ldots,m).$$

Finally, $\Pi_m = 1/\Gamma_m$ what is in agreement with (1).

Remark 2. The expectation of the number of the busy channels at a moment when a call occurs, is (cf. [13])

$$\mathbf{E}\{\eta_m^*\} = \sum_{j=0}^{m} j P_j = \frac{\varphi_1}{1-\varphi_1}(1-\Pi_m).$$

If we define the process $\{\eta(t)\}$ for all t, then it can be shown that the limiting distribution $\lim_{t\to\infty} \mathbf{P}\{\eta(t)=j\} = P_j^*$ $(j=0,1,\ldots,m)$ exists when $F(x)$ is not a lattice type distribution and $\alpha = \int_0^\infty x\,dF(x) < \infty$. The distribution $\{P_j^*\}$ is independent of the initial distribution of $\eta(0)$. (Cf. [13].) In this case we can define a stationary process supposing that $\mathbf{P}\{\eta^*(0)=j\} = P_j^*$ $(j=0,1,\ldots,m)$. For a stationary process $\{\eta^*(t)\}$ the expectation of the number of the busy channels at an arbitrary instant t is

$$\mathbf{E}\{\eta^*(t)\} = \sum_{j=0}^{m} j P_j^* = \frac{1-\Pi_m}{\mu\alpha},$$

which can be shown also in a heuristic way.

4. A connected servicing problem. Suppose that $m+1$ automatic machines are serviced by a single repairman. The machines are working continuously, however, at any time a machine may break down and call for service. We suppose that if at time t a machine is working, the probability that it will call for service in the time interval $(t, t+\Delta t)$ is $\mu\Delta t + o(\Delta t)$ for each machine. We assume that the machines work independently. If a machine breaks down it will be serviced immediately unless the repairman is servicing another machine, in which case a waiting line is formed. Assume that the repairman is working if there is at least one machine in the waiting line. Suppose that the times required for servicing of the machines (service times) are indentically distributed independent positive random variables with distribution function

$F(x)$. Let us denote by $\xi(t)$ the number of machines working at the instant t. Denote by $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ the endpoints of the consecutive service times. Put $\xi(\tau_n - 0) = \xi_n$. The random variables $\{\xi_n\}$ $(n = 1, 2, 3, \ldots)$ form a Markov chain and there exists a limiting distribution $\lim_{n \to \infty} \mathbf{P}\{\xi_n = j\} = P_j$ $(j = 0, 1, \ldots, m)$ which is independent of the initial distribution of ξ_1. According to an earlier work [14] of the author, it can be established that the distribution $\{P_j\}$ is exactly the same which has been considered in Section 1. Now if we define also a stationary Markov chain $\{\xi_n^*\}$, then this Markov chain shows the same stochastic behaviour as the chain $\{\eta_n^*\}$ defined in Section 1.

Denote by G_{m+1} the expectation of the number of services in a service period. Evidently we have $G_{m+1} = 1/P_m$, i. e. G_{m+1} agrees with Γ_m defined in Section 3. The explicit form of G_{m+1} was determined by H. ASHCROFT [1] and earlier by R. KRONIG and H. MONDRIA [7].

The mentioned servicing problem was treated first by A. JA. KHINTCHINE [5], but he did not give an explicit solution. The particular case when $F(x)$ is an exponential distribution function was solved by several authors (cf. W. FELLER [3], p. 379).

REMARK 3. If we define the process $\{\xi(t)\}$ for all t, then it can be shown that the limiting distribution $\lim_{t \to \infty} \mathbf{P}\{\xi(t) = j\} = Q_j^*$ $(j = 0, 1, \ldots, m+1)$ exists when $\alpha = \int_0^\infty x\,dF(x) < \infty$. The distribution $\{Q_j^*\}$ is independent of the initial distribution of $\xi(0)$ (cf. [14]). In this case it is possible to define a stationary process $\{\xi^*(t)\}$. For the stationary process the expected number of the machines working at an arbitrary instant t is

$$\mathbf{E}\{\xi^*(t)\} = \sum_{j=0}^{m+1} j Q_j^* = \frac{(m+1)}{P_m + (m+1)\alpha\mu} = \frac{(m+1)\,G_{m+1}}{1 + (m+1)\alpha\mu G_{m+1}},$$

which can be shown also in a heuristic way.

5. Remarks on Erlang's formula. If, in particular, $\{\tau_n\}$ forms a Poisson process with density λ, then by (1) we obtain

$$\Pi_m = \frac{\dfrac{1}{m!}\left(\dfrac{\lambda}{\mu}\right)^m}{\displaystyle\sum_{j=0}^{m} \frac{1}{j!}\left(\frac{\lambda}{\mu}\right)^j}$$

which is the well-known ERLANG's formula.

It is to be remarked that the above formula is valid also in the case when $\{\tau_n\}$ forms a Poisson process with density λ and $H(x)$ is an arbitrary distribution function with mean $1/\mu$.

I take the opportunity to correct my Remark 1 in [13]. When I was writing my paper [13], then I believed that under general conditions the proof of ERLANG's formula is still missing. I am indebted to Mr. R. SYSKI who kindly called my attention that earlier F. POLLACZEK [10], C. PALM [8], L. KOSTEN [6] and others have given proofs for ERLANG's formula. Further I should like to remark that recently B. A. SEVASTJANOV [12] has given an exact proof for ERLANG's formula.

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ON A QUEUEING PROBLEM CONCERNING TELEPHONE TRAFFIC

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(*Presented by* A. Rényi)

Dedicated to the memory of A. K. Erlang *on the occasion of the anniversary of his 80th birthday, 1st January 1958*

1. Summary. Let us suppose that at a telephone center calls are arriving in the instants $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ $(0 < \tau_1 < \tau_2 < \cdots < \tau_n < \infty)$. There are m available channels. Suppose that a connection (conversation) is realised if the incoming call finds an idle channel. If all channels are busy, each new call joins a waiting line and waits until a channel is freed. (Waiting system.)

The following assumptions are made:

A) The instants $\{\tau_n\}$ form a sequence of recurrent events, i. e. the time differences $\tau_{n+1} - \tau_n$ $(n = 1, 2, 3, \ldots)$ are identically distributed independent positive random variables. Denote by $F(x)$ their common distribution function. Suppose that

$$\alpha = \int_0^\infty x\, d F(x) < \infty$$

and put

$$\varphi(s) = \int_0^\infty e^{-sx}\, d F(x)$$

for non-negative real s.

B) Denote by χ_n the duration of the connection which is realised by the call arriving at the instant τ_n. It is assumed that the χ_n $(n = 1, 2, 3, \ldots)$ are identically distributed independent positive random variables which are independent also of the process $\{\tau_n\}$, and that they have the following distribution function:

$$\mathbf{P}\{\chi_n \leqq x\} = \begin{cases} 1 - e^{-\mu x} & \text{if } x \geqq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Denote by $\eta(t)$ the total number of the busy channels and the calls in the waiting line at the instant t and put $\eta(\tau_n - 0) = \eta_n$ $(n = 1, 2, 3, \ldots)$. We say that the system is in state E_k $(k = 0, 1, 2, \ldots)$ at the instant t if $\eta(t) = k$. If the system is in a state E_k where $k \leqq m$, then k connections are going on; however, if $k > m$, then only m connections are going on and there are $k - m$ calls in the waiting line.

We shall determine the limiting distribution $\lim_{n\to\infty} \mathbf{P}\{\eta_{\tau_n}=k\}=P_k$ $(k=0, 1, 2, \ldots)$ which exists independently of the initial state if $m\alpha\mu>1$. Further we shall show that if $m\alpha\mu>1$ and $F(x)$ is not a lattice distribution, then there exists the limiting distribution $\lim_{t\to\infty} \mathbf{P}\{\eta(t)=k\}=P_k^*$ independently of the initial state. Next we shall define a stationary process $\{\eta(t)\}$ for which $\mathbf{P}\{\eta(t)=k\}=P_k^*$ for all t $(0\leqq t<\infty)$, and we determine $G^*(x)$, the distribution function of the waiting time for an arbitrary call.

2. Introduction. Several authors (A. K. Erlang [2], E. C. Molina [8] and A. N. Kolmogorov [6]) treated the above problem in the particular case when the incoming calls form a Poisson process, i. e. $F(x)=1-e^{-\lambda x}$ if $x\geqq 0$. The general case when $F(x)$ is arbitrary, was treated by D. G. Kendall [5]. Kendall proved the existence of the limiting distribution $\{P_k\}$ when $m\alpha\mu>1$, but he did not determine the explicit form of $\{P_k\}$. He showed only that the probabilities $P_m, P_{m+1}, P_{m+2}, \ldots$ form a geometric series with known quotient. Further he proved that $G^*(x)$ is an exponential distribution with an indefinite parameter.

Finally, we remark that in paper [9] we dealt with an analogous problem supposing that an incoming call is lost if all channels are busy. (Busy-signal system.)

3. The determination of the limiting distribution $\{P_k\}$. We shall prove the following

Theorem 1. *If $m\alpha\mu>1$, then the limiting distribution $\lim_{n\to\infty}\mathbf{P}\{\eta_{\tau_n}=k\}=P_k$ $(k=0, 1, 2, \ldots)$ exists and is independent of the initial state. We have*

$$P_k=\begin{cases}\displaystyle\sum_{r=k}^{m-1}(-1)^{r-k}\binom{r}{k}U_r & (k=0, 1, \ldots, m-1),\\ A\,\omega^{k-m} & (k=m, m+1, \ldots).\end{cases} \tag{1}$$

Here ω is the only root of the equation

$$\int_0^\infty e^{-m\mu(1-\omega)x}\,dF(x)=\omega \tag{2}$$

for which $0<\omega<1$ and

$$U_r=A\,C_r\sum_{j=r+1}^{m}\frac{\binom{m}{j}}{C_j(1-\varphi_j)}\left[\frac{m(1-\varphi_j)-j}{m(1-\omega)-j}\right] \tag{3}$$

where

$$A=\frac{1}{\dfrac{1}{1-\omega}+\displaystyle\sum_{j=1}^{m}\frac{\binom{m}{j}}{C_j(1-\varphi_j)}\left[\frac{m(1-\varphi_j)-j}{m(1-\omega)-j}\right]} \tag{4}$$

with the abbreviations

(5) $$C_j = \frac{\varphi_1}{1-\varphi_1} \cdot \frac{\varphi_2}{1-\varphi_2} \cdots \frac{\varphi_j}{1-\varphi_j}$$

and

(6) $$\varphi_j = \int_0^\infty e^{-j\mu x}\, dF(x).$$

If we introduce

(7) $$B_r = \sum_{k=r}^{\infty} \binom{k}{r} P_k,$$

the r-th binomial moment of $\{P_k\}$, *then we have*

(8) $$B_r = \begin{cases} U_r + \dfrac{A}{1-\omega} \displaystyle\sum_{j=0}^{r} \binom{m}{j} \left(\frac{\omega}{1-\omega}\right)^{r-j} & \text{if} \quad r < m, \\[2ex] \dfrac{A\,\omega^{r-m}}{(1-\omega)^{r+1}} & \text{if} \quad r \geqq m. \end{cases}$$

PROOF. It is easy to see that the sequence of random variables $\{\eta_n\}$ $(n = 1, 2, 3, \ldots)$ forms a Markov chain with transition probabilities $\mathbf{P}\{\eta_{n+1} = k \mid \eta_n = j\} = p_{jk}$, where

$$p_{jk} = \binom{j+1}{k} \int_0^\infty e^{-k\mu x} (1 - e^{-\mu x})^{j+1-k}\, dF(x) \quad \text{if} \quad j < m,$$

(9) $$p_{jk} = \binom{m}{k} \int_0^\infty e^{-k\mu x} \left[\int_0^x \frac{(m\mu y)^{j-m}}{(j-m)!} (e^{-\mu y} - e^{-\mu x})^{m-k} m\mu\, dy \right] dF(x)$$

$$\text{if} \quad j \geqq m \quad \text{and} \quad k < m,$$

$$p_{jk} = \int_0^\infty e^{-m\mu x} \frac{(m\mu x)^{j+1-k}}{(j+1-k)!}\, dF(x) \quad \text{if} \quad j \geqq m \quad \text{and} \quad k \geqq m.$$

Under the condition that the inter-arrival times are constant and equal to x let us denote by $\pi_{jk}(x)$ the transition probabilities. Then in the general case

$$p_{jk} = \int_0^\infty \pi_{jk}(x)\, dF(x).$$

If we suppose that $m\alpha\mu > 1$, then it can be shown that the Markov chain $\{\eta_n\}$ is ergodic. The details of the proof can be found in the work [5] of D. G. KENDALL. Thus the limiting probabilities $\lim\limits_{n\to\infty} \mathbf{P}\{\eta_n = k\} = P_k$ $(k = 0, 1, 2, \ldots)$ exist and are independent of the initial distribution of η_1. The limiting distribution $\{P_k\}$ is uniquely determined by the following system

of linear equations:

(10) $$P_k = \sum_{j=k-1}^{\infty} p_{jk} P_j$$

and

$$\sum_{k=0}^{\infty} P_k = 1$$

(cf. W. FELLER [3], p. 325).

First of all following KENDALL let us consider the equations (10) for $k \geqq m$. So we have

$$P_k = \sum_{\nu=0}^{\infty} P_{\nu+k-1} \int_0^{\infty} e^{-m\mu x} \frac{(m\mu x)^\nu}{\nu!} dF(x) \qquad \text{if} \qquad k \geqq m.$$

Choose the positive real $\omega \neq 1$ such that

$$\int_0^{\infty} e^{-m\mu(1-\omega)x} dF(x) = \omega.$$

If $m\alpha\mu > 1$, then there exists only one such ω with $0 < \omega < 1$. Namely, $\varphi(0) = 1$, $\varphi'(0) = -\alpha$ and $\varphi(s)$ is monotone decreasing if $0 \leqq s < \infty$.

Now if we suppose

(11) $$P_k = A\,\omega^{k-m} \qquad (k \geqq m),$$

then we see that the equations (10) are satisfied if $k > m$, and if $k = m$, then we obtain $P_{m-1} = A/\omega$. Hence $A = P_{m-1}\omega$. It remains only to determine the unknown probabilities $P_0, P_1, \ldots, P_{m-1}$. Let us introduce the following generating function:

$$U(z) = \sum_{k=0}^{m-1} P_k z^k.$$

Then from (10) we obtain

(12) $$U(z) = \int_0^{\infty} (1 - e^{-\mu x} + z e^{-\mu x})\, U(1 - e^{-\mu x} + z e^{-\mu x})\, dF(x) +$$
$$+ A \int_0^{\infty} \left[\int_0^{x} e^{m\mu\omega y} (e^{-\mu y} - e^{-\mu x} + z e^{-\mu x})^m m\mu\, dy \right] dF(x) - A z^m.$$

By (11)

(13) $$U(1) = \sum_{k=0}^{m-1} P_k = 1 - \sum_{k=m}^{\infty} P_k = 1 - \frac{A}{1-\omega}.$$

Let us introduce the following notation:

$$U_j = \frac{1}{j!} \left(\frac{d^j U(z)}{dz^j} \right)_{z=1} \qquad (j = 0, 1, \ldots, m-1).$$

Then we obtain by (13)

$$U_0 = 1 - \frac{A}{1-\omega}, \tag{14}$$

and by j-times differentiating (12) and putting $z=1$ we obtain for $j=1, 2, \dots, m-1$

$$U_j = U_j \varphi_j + U_{j-1}\varphi_j - A\binom{m}{j}\frac{m(1-\varphi_j)-j}{m(1-\omega)-j}$$

using $\varphi(m\mu(1-\omega)) = \omega$. Thus

$$U_j = \frac{\varphi_j}{1-\varphi_j}U_{j-1} - \frac{A\binom{m}{j}}{(1-\varphi_j)}\left[\frac{m(1-\varphi_j)-j}{m(1-\omega)-j}\right]. \tag{15}$$

This is a linear difference equation of first order with variable coefficients which can be solved easily (cf. CH. JORDAN [4], p. 583). Define $C_0 = 1$ and

$$C_j = \frac{\varphi_1}{1-\varphi_1}\cdot\frac{\varphi_2}{1-\varphi_2}\cdots\frac{\varphi_j}{1-\varphi_j}$$

and divide both sides of (15) by C_j, then we obtain

$$\frac{U_j}{C_j} = \frac{U_{j-1}}{C_{j-1}} - \frac{A\binom{m}{j}}{C_j(1-\varphi_j)}\cdot\left[\frac{m(1-\varphi_j)-j}{m(1-\omega)-j}\right].$$

Summing up this formula for $j = r+1, \dots, m-1$ and taking into consideration that $U_{m-1} = P_{m-1} = A/\omega$, we obtain for $r = 0, 1, 2, \dots, m-1$

$$\frac{U_r}{C_r} = A\sum_{j=r+1}^{m}\frac{\binom{m}{j}}{C_j(1-\varphi_j)}\left[\frac{m(1-\varphi_j)-j}{m(1-\omega)-j}\right]. \tag{16}$$

If $r=0$, then by (14) $\frac{U_0}{C_0} = 1 - \frac{A}{1-\omega}$ and comparing this with (16) for $r=0$ we obtain

$$A = \frac{1}{\dfrac{1}{1-\omega} + \displaystyle\sum_{j=1}^{m}\frac{\binom{m}{j}}{C_j(1-\varphi_j)}\left[\frac{m(1-\varphi_j)-j}{m(1-\omega)-j}\right]}$$

what proves (4).

Thus $U(z)$ is uniquely determined. The unknown probabilities P_k $(k = 0, 1, \dots, m-1)$ can be expressed as

$$P_k = \frac{1}{k!}\left(\frac{d^k U(z)}{dz^k}\right)_{z=0}.$$

The derivatives

$$U_r = \frac{1}{r!}\left(\frac{d^r U(z)}{dz^r}\right)_{z=1} \qquad (r=0,1,\ldots,m-1)$$

are known and by means of these we can get easily that

$$P_k = \sum_{r=k}^{m-1}(-1)^{r-k}\binom{r}{k}U_r \qquad (r=0,1,\ldots,m-1)$$

what proves (1).

The binomial moments B_r $(r=0,1,2,\ldots)$ can be determined by the aid of the generating function

$$U^*(z)=\sum_{k=0}^{\infty}P_k z^k = U(z)+\frac{Az^m}{1-\omega z}.$$

Hence

$$B_r = \frac{1}{r!}\left(\frac{d^r U^*(z)}{dz^r}\right)_{z=1} = \sum_{k=r}^{\infty}\binom{k}{r}P_k = U_r + \frac{A}{1-\omega}\sum_{j=0}^{r}\binom{m}{j}\left(\frac{\omega}{1-\omega}\right)^{r-j}$$

where $U_r=0$ if $r\geq m$. This proves (8).

Example. Let $\{\tau_n\}$ be a Poisson process with intensity λ. Then

$$F(x)=\begin{cases}1-e^{-\lambda x} & \text{if } x\geq 0,\\ 0 & \text{if } x<0,\end{cases}$$

$\alpha = 1/\lambda$ and $\varphi(s)=\lambda/(\lambda+s)$. If $\lambda < m\mu$, then there exists a limiting distribution $\{P_j\}$. Since by (2) we obtain $\omega=\lambda/m\mu$, consequently

$$P_j=\begin{cases}A\dfrac{m!}{j!}\left(\dfrac{\lambda}{\mu}\right)^{j-m} & (j=0,1,\ldots,m-1),\\[2ex] A\left(\dfrac{\lambda}{m\mu}\right)^{j-m} & (j=m,m+1,\ldots)\end{cases}$$

where

$$A=\frac{\dfrac{1}{m!}\left(\dfrac{\lambda}{\mu}\right)^m}{\displaystyle\sum_{j=0}^{m-1}\frac{1}{j!}\left(\frac{\lambda}{\mu}\right)^j+\frac{1}{m!}\left(\frac{\lambda}{\mu}\right)^m\frac{1}{1-\dfrac{\lambda}{m\mu}}},$$

namely in (4)

$$\frac{m(1-\varphi_j)-j}{m(1-\omega)-j}=\frac{j\mu}{\lambda+j\mu}$$

and

$$C_j=\prod_{k=1}^{j}\frac{\varphi_k}{1-\varphi_k}=\frac{1}{j!}\left(\frac{\lambda}{\mu}\right)^j.$$

4. The determination of the limiting distribution $\{P_k^*\}$. So long we did not deal with the role of the instant τ_1 of the first call in the time interval $(0, \infty)$. Now let us suppose that τ_1 is an arbitrary random variable which is independent of the other random variables in question.

THEOREM 2. *If $m\alpha\mu > 1$ and $F(x)$ is not a lattice distribution, then the limiting distribution $\lim_{t\to\infty} \mathbf{P}\{\eta(t) = k\} = P_k^*$ $(k = 0, 1, 2, \ldots)$ exists and is independent of the initial state. We have*

$$P_0^* = 1 - \frac{1}{m\alpha\mu} - \frac{1}{\alpha\mu}\sum_{j=1}^{m-1} P_{j-1}\left(\frac{1}{j} - \frac{1}{m}\right),$$

(17)
$$P_k^* = \frac{P_{k-1}}{k\alpha\mu} \quad \textit{if} \quad k = 1, 2, \ldots, m-1,$$

$$P_k^* = \frac{P_{k-1}}{m\alpha\mu} \quad \textit{if} \quad k = m, m+1, \ldots.$$

We need two lemmas.

LEMMA 1. *Denote by $M_k(t)$ the expectation of the number of transitions $E_k \to E_{k+1}$ $(k = 0, 1, 2, \ldots)$ occurring in the time interval $(0, t]$. If $m\alpha\mu > 1$ and $F(x)$ is not a lattice distribution, then for all $h > 0$ we have*

(18)
$$\lim_{t\to\infty} \frac{M_k(t+h) - M_k(t)}{h} = \frac{P_k}{\alpha}.$$

PROOF. The time differences between consecutive transitions $E_k \to E_{k+1}$ are, as it can easily be seen, identically distributed independent positive random variables. If $F(x)$ is not a lattice distribution, then these random variables are not lattice distributed either. If $m\alpha\mu > 1$, then these random variables have a finite expectation α/P_k. Under these conditions according to an easy extension of a theorem stated by D. BLACKWELL [1], it follows that (18) holds and this limit is independent of the initial value $\eta(0)$. It remains to prove that the expectation in question is α/P_k. Let us consider the Markov chain $\{\eta_n\}$. The state E_k is a recurrent state and the expectation of the recurrence step number is $1/P_k$ (cf. W. FELLER [3], p. 325). As transitions $E_k \to E_{k+1}$ occur only at such instants τ_n $(n = 1, 2, 3, \ldots)$ for which $\eta_n = k$, consequently the expected number of steps between consecutive transitions $E_k \to E_{k+1}$ is $1/P_k$. The expectation of the length of each step is α and so using the Markov character it follows by the aid of the known theorem of A. WALD (cf. A. N. KOLMOGOROV and YU. V. PROHOROV [7]) that the expectation of the time differences between consecutive transitions $E_k \to E_{k+1}$ is α/P_k.

Now we can write

$$\mathbf{P}\{\eta_t(t)=k\}=\sum_{j=k-1}^{\infty}\int_0^t \pi_{jk}(t-u)\,[1-F(t-u)]\,dM_j(u). \tag{19}$$

Namely, the event $\eta_t(t)=k$ can occur in the following mutually exclusive ways: at the instant u (where $0\leq u\leq t$) there occurs a transition $E_j\rightarrow E_{j+1}$ ($j=k-1,k,\ldots$) and the next call occurs after the instant t (the probability of that is $1-F(t-u)$) and in the time interval (u,t) $j+1-k$ connections terminate and k connections do not terminate (the probability of that is $\pi_{jk}(t-u)$). Finally, (19) follows by the total probability theorem.

Using (18) we obtain from (19) by estimating the lower and upper approximative sums of the integral that $\lim\limits_{t\rightarrow\infty}\mathbf{P}\{\eta_t(t)=k\}=P_k^*$ exists and it is independent of the initial distribution of $\eta_t(0)$. We obtain

$$P_k^*=\sum_{j=k-1}^{\infty} p_{jk}^* P_j \tag{20}$$

where

$$p_{jk}^*=\frac{1}{\alpha}\int_0^{\infty}\pi_{jk}(x)\,[1-F(x)]\,dx. \tag{21}$$

So we obtained an explicit form for the distribution $\{P_k^*\}$ but it can be given also in a simpler way.

LEMMA 2. *Denote by $N_k(t)$ the expectation of the number of transitions $E_k\rightarrow E_{k+1}$ $(k=1,2,3,\ldots)$ occurring in the time interval $(0,t]$. If $m\alpha\mu>1$ and $F(x)$ is not a lattice distribution, then we have*

$$\lim_{t\rightarrow\infty}N_k'(t)=\begin{cases}k\mu P_k^* & \text{if } k\leq m,\\ m\mu P_k^* & \text{if } k\geq m.\end{cases} \tag{22}$$

It is easy to see that

$$N_k(t+\Delta t)=N_k(t)+\mathbf{P}\{\eta_t(t)=k\}\,k\mu\,\Delta t+o(\Delta t)\quad \text{if}\quad k\leq m$$

and

$$N_k(t+\Delta t)=N_k(t)+\mathbf{P}\{\eta_t(t)=k\}\,m\mu\,\Delta t+o(\Delta t)\quad \text{if}\quad k\geq m.$$

As $\lim\limits_{t\rightarrow\infty}\mathbf{P}\{\eta_t(t)=k\}=P_k^*$ exists, (22) follows.

PROOF OF THEOREM 2. Since the difference of the number of transitions $E_{k-1}\rightarrow E_k$ and $E_k\rightarrow E_{k+1}$ occurring in the time interval $(0,t]$ is at most 1, it holds

$$|M_{k-1}(t)-N_k(t)|\leq 1. \tag{23}$$

As from (18)

$$\lim_{t\to\infty} \frac{M_{k-1}(t)}{t} = \frac{P_{k-1}}{\alpha}$$

and from (22)

$$\lim_{t\to\infty} \frac{N_k(t)}{t} = \begin{cases} k\mu P_k^* & \text{if} \quad k \leq m, \\ m\mu P_k^* & \text{if} \quad k \geq m, \end{cases}$$

we obtain (17) by (23) where particularly $P_0^* = 1 - \sum_{j=1}^{\infty} P_j^*$.

5. The stationary process. Denote by $\zeta(t)$ the distance between the instant t and the instant of the next call. The conditional limiting distributions are described by the following

THEOREM 3. *If $m\alpha\mu > 1$ and the distribution function $F(x)$ is not of lattice type, then the limiting distribution*

(24) $$\lim_{t\to\infty} \mathbf{P}\{\zeta(t) \leq x \mid \eta(t) = k\} = F_k^*(x)$$

exists and we have

(25) $$F_k^*(x) = \frac{1}{\alpha P_k^*} \sum_{j=k-1}^{\infty} P_j \int_0^{\infty} [F(x+y) - F(y)]\, \pi_{jk}(y)\, dy.$$

PROOF. We have

(26) $$\mathbf{P}\{\zeta(t) \leq x, \eta(t) = k\} = \sum_{j=k-1}^{\infty} \int_0^t \pi_{jk}(t-u)[F(t+x-u) - F(t-u)]\, dM_j(u).$$

Namely, the event $\{\zeta(t) \leq x, \eta(t) = k\}$ can occur if at some instant u (where $0 \leq u \leq t$) a transition $E_j \to E_{j+1}$ $(j = k-1, k, \ldots)$ takes place and the next call occurs in the time interval $(t, t+x)$, further during the time interval (u, t) k connections do not come to an end while the others come to an end. So we obtain (26) by the aid of the total probability theorem. Since

$$\mathbf{P}\{\zeta(t) \leq x \mid \eta(t) = k\} = \frac{\mathbf{P}\{\zeta(t) \leq x, \eta(t) = k\}}{\mathbf{P}\{\eta(t) = k\}}$$

and $\lim_{t\to\infty} \mathbf{P}\{\eta(t) = k\} = P_k^*$ exists, we obtain (24).

REMARK 1. If the distribution function $F(x)$ is not of lattice type, then we have

(27) $$\lim_{t\to\infty} \mathbf{P}\{\zeta(t) \leq x\} = F^*(x)$$

where

(28) $$F^*(x)=\begin{cases}\dfrac{1}{\alpha}\displaystyle\int_0^x[1-F(y)]\,dy & \text{if} \quad x\geqq 0,\\ 0 & \text{if} \quad x<0.\end{cases}$$

This is an easy consequence of BLACKWELL's theorem.

The investigated process may be considered as a Markov process if we describe the state of the system by the pair of random variables $\{\eta_t(t), \xi(t)\}$. If we suppose that $m\alpha\mu<1$ and the initial distribution of the process $\{\eta_t(t), \xi(t)\}$ $(0\leqq t<\infty)$ is given by $\mathbf{P}\{\eta_t(0)=k\}=P_k^*$ and $\mathbf{P}\{\xi(0)\leqq x\,|\,\eta_t(0)=k\}=F_k^*(x)$ $(k=0,1,2,\ldots)$, then we obtain the stationary process. In case of the stationary process we have for all t that $\mathbf{P}\{\eta_t(t)=k\}=P_k^*$, and for all n that $\mathbf{P}\{\eta_{t_n}=k\}=P_k$. The first statement is evident. To prove the second it is sufficient to show that

(29) $$P_k=\sum_{j=k-1}^{\infty}P_j^*\int_0^{\infty}\pi_{jk}(x)\,dF_j^*(x)$$

which can be proved by the aid of (25).

6. The distribution function of the waiting time. If $m\alpha\mu<1$, then let us consider the stationary process. Denote by $G^*(x)$ the distribution function of the waiting time of an arbitrary call, provided that the connections are performed in the order of the arrival of the calls.

THEOREM 4. *For the stationary process the distribution function of the waiting time of an arbitrary call is*

(30) $$G^*(x)=1-\frac{A\,e^{-m\mu(1-\omega)x}}{(1-\omega)}\qquad (x\geqq 0),$$

if the connections are performed in the order of the arrival of the calls. Here ω and A have the same meaning as in Section 3.

PROOF. We can write easily that

$$G^*(x)=\sum_{j=0}^{m-1}P_j+\sum_{j=m}^{\infty}P_j\int_0^x e^{-m\mu y}\frac{(m\mu y)^{j-m}}{(j-m)!}\,m\mu\,dy.$$

Namely, if a call happens, then the system is in state E_j with probability P_j. If $j<m$, then there is no waiting time. If $j\geqq m$, then the call must wait for $j+1-m$ successive terminations of the connections and the endpoints of these connections form a Poisson process with density $m\mu$. Since $P_j=A\omega^{j-m}$ if $j\geqq m$ and $\sum_{j=0}^{m-1}P_j=U_0=1-\dfrac{A}{1-\omega}$, we obtain (30).

REMARK 2. The expectation of the waiting time is

$$\Gamma^* = \int_0^\infty [1-G^*(x)]\,dx = \frac{A}{m\mu(1-\omega)^2}$$

and this is independent of the order of connections corresponding to the calls.

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ON STOCHASTIC SET FUNCTIONS. II

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Introduction

The present paper contains an outline of the stochastic integral which can be defined relative to a completely additive stochastic set function $\xi(A)$ $(A \in \mathfrak{S})$.

Many types of stochastic integrals are known in the probability theory. Historically the first one is due to N. WIENER [21]. This was generalized by K. ITÔ by omitting the condition that the integrand is a constant function [10] and in another direction by considering a multidimensional Wiener process instead of the simple one [11]. An important step in the development was the integral representation of stochastic processes with independent increments. This result is essentially due to P. LÉVY [16] but a rigorous and complete treatment was given by K. ITÔ [9] (see also [12]). A stochastic Stieltjes integral with respect to a stochastic process with independent increments is used in [12]. The stochastic integral occurring in the theory of stationary processes was introduced by A. N. KOLMOGOROV [14] [15] and discussed from a general point of view by A. OBUKHOV [17]. These integrals were later generalized by J. L. DOOB [5]. Another generalization is due to S. BOCHNER [1] for the case of an abstract space and for an *additive* random set function. Recently V. FABIAN [6] has defined a stochastic integral with a non-negative stochastic measure.

The generality of our integral introduced in § 1 of Chapter I is contained in the following: The space where we integrate is abstract and we do not suppose at all the existence of any moments. This makes our integral of new type since in the above-mentioned stochastic integrals, except that of V. FABIAN who uses the non-negativity of the random measure, the existence of at least one moment of order $p \geq 1$ is always supposed. The speciality of the integral (1. 3) is that the stochastic set function $\xi(A)$ $(A \in \mathfrak{S})$ has the following property: to disjoint sets $A_1, \ldots, A_n$ there correspond independent random variables. But exactly this property makes possible the proof of the existence and many properties of our integral.

We introduce a weaker and a stronger form of our integral. That one introduced by Definition 4, the u-integral is a generalization of the Radon integral and reduces to the latter if the random variables $\xi(A)$ $(A \in \mathfrak{S})$ are constants with probability 1. Most of the properties which are true in case of the Radon integral remain true in case of the u-integral too.

In the formulation of the theorems we did not take into consideration those trivial generalizations in which the difference from the original theorem is the neglect of some 0-sets. This generalizations can be formulated without any difficulty.

The whole theory of our integral is based on the following property of completely additive stochastic set functions: the set of the distribution functions $\{F(x, A), A \in \mathfrak{S}\}$ is (conditionally) compact in the space of the one-dimensional distribution functions. This theorem is proved in [18] (Chapter III, Theorem 3. 4).

To the practical applications we shall return later.

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Definitions and notations

We keep all the notions and notations introduced in [18] (pp. 217—218). In the whole paper H denotes the basic space where we integrate, $\mathfrak{S}$ a σ-ring of some subsets of H and $\xi(A)$ $(A \in \mathfrak{S})$ a completely additive set function with respect to which we integrate. The variable element of H will be denoted by h. Only in Chapter III will be considered integrals with respect to more than one set functions, and in Chapter IV in some statements we replace the σ-ring $\mathfrak{S}$ by a ring $\mathfrak{R}$.

If we say that $\varphi(h)$ is integrable, we mean that it is integrable with respect to the completely additive set function $\xi(A)$ $(A \in \mathfrak{S})$. The terminology "$\varphi(h)$ is integrable" includes the measurability of the function.

We have to introduce only one new notion which did not take place in [18]. This is the following: a set $X \in \mathfrak{S}$ is said to be a 0-set with respect to the completely additive set function $\xi(A)$ $(A \in \mathfrak{S})$ if for every $Y \in X\mathfrak{S}$ we have $\xi(Y) = 0$.

I. DEFINITION AND EXISTENCE OF THE INTEGRAL

§ 1. Definitions

In order to simplify our expressions we introduce the following definitions:

DEFINITION 1. A sequence of real numbers $\{y_k\}$ which is infinite in two directions will be called a *dividing point sequence* if $y_k < y_{k+1}$ $(k=0, \pm 1, \pm 2, \ldots)$, $\sup\limits_k (y_{k+1}-y_k) < \infty$, $\lim\limits_{k\to\infty} y_k = \infty$ and $\lim\limits_{k\to-\infty} y_k = -\infty$. A sequence of dividing point sequences $\{y_k^{(n)}\}$ will be called a *dividing point double sequence.*

DEFINITION 2. A dividing point double sequence $\{y_k^{(n)}\}$ will be called *infinitely fine* if

$$\lim_{n\to\infty} \sup_k (y_{k+1}^{(n)}-y_k^{(n)}) = 0.$$

Let $\varphi(h)$ be a real function defined on the elements of H and measurable with regard to the σ-ring $\mathfrak{S}$. Let furthermore $\{y_k\}$ be a dividing point sequence and consider the series of independent random variables

$$(1.1) \qquad \sum_{k=-\infty}^{\infty} y_k \xi(AH_k),$$

where $A \in \mathfrak{S}$, $H_k = \{h: y_k < \varphi(h) \leq y_{k+1}\}$. The sets H_k are disjoint and their union equals H.

The system of sets $\{H_k\}$ will be called the *subdivision* corresponding to the function $\varphi(h)$ and the dividing point sequence $\{y_k\}$. Now we formulate the definition of the integral.

DEFINITION 3. Let $\varphi(h)$ $(h \in H)$ be a measurable function and suppose that there exists a $\delta > 0$ such that for every dividing point sequence $\{y_k\}$ with $\sup\limits_k (y_{k+1}-y_k) \leq \delta$ the series (1. 1) converges with probability 1 regardless of the order of summation. Let furthermore $\{y_k^{(n)}\}$ be an infinitely fine dividing point double sequence and put

$$(1.2) \qquad \eta_n(A) = \sum_{k=-\infty}^{\infty} y_k^{(n)} \xi(AH_k^{(n)}).$$

If the sequence (1. 2) converges stochastically to a limiting random variable $\eta(A)$, then $\varphi(h)$ will be called integrable over the set A. The random variable $\eta(A)$, which is obviously independent of the special choice of $\{y_k^{(n)}\}$, will

be called the integral of $\varphi(h)$ over the set A and will be denoted by

$$\int_A \varphi(h)\xi(dA). \tag{1.3}$$

We shall prove (Theorem 1.3) that if (1.1) converges for every sufficiently fine dividing point sequence, then the integral (1.3) exists. The integral introduced by Definition 3 has, however, some undesirable properties. First of all we mention that the classical Radon integral is not a special case of this. For instance, the function $\psi(h)$ defined by (2.19) is not integrable in the sense of Radon with respect to the generalized measure (2.18), but is integrable in the sense of Definition 3. This implies other unpleasant properties as follows: if $\varphi(h)$ and $\psi(h)$ are measurable functions, $\psi(h)$ is integrable over the set A and $|\varphi(h)| \leq \psi(h)$, then $\varphi(h)$ is not necessarily integrable over A. The same holds for the sum of two integrable functions $\varphi_1(h), \varphi_2(h)$. Both cases are illustrated by examples in Chapter II, § 3.

In order to avoid the above-mentioned anomalies we introduce another integral supposing more on the function $\varphi(h)$.

Definition 4. A measurable function $\varphi(h)$ will be called *unconditionally integrable* (*u-integrable*) over the set $A \in \mathfrak{S}$ if it is integrable in the sense of Definition 3 over all measurable subsets of A.

Now, the u-integral is a generalization of the Radon integral. In fact, if the stochastic set function $\xi(A)$ $(A \in \mathfrak{S})$ reduces to a number-valued completely additive set function $\mu(A)$ $(A \in \mathfrak{S})$, then, according to the decomposition theorem of Hahn, every set $A \in \mathfrak{S}$ is the sum of two disjoint measurable sets A_1, A_2 with the property that

$$\mu(B) \geq 0 \quad \text{for} \quad B \in A_1\mathfrak{S},$$
$$\mu(B) \leq 0 \quad \text{for} \quad B \in A_2\mathfrak{S}.$$

The u-integrability of $\varphi(h)$ implies the existence of the Lebesgue integrals

$$\int_{A_1} \varphi(h)\mu(dA), \qquad \int_{A_2} \varphi(h)(-\mu(dA)),$$

hence our assertion follows.

It will be shown that the u-integral has analogous properties to the Radon integral.

We anticipate a special case of Theorem 2.7. If $\varphi(h)$ is measurable and $|\varphi(h)|$ is u-integrable over $A \in \mathfrak{S}$, then the same holds for $\varphi(h)$. The conversion is also true. In fact, if $\varphi(h)$ is measurable and u-integrable, then the positive and negative parts of $\varphi(h)$ are u-integrable, hence, using Theorem 2.2, the statement follows.

§ 2. Two auxiliary theorems

In the paper [18] we have introduced the following set function:

$$W(T, A) = \mathrm{Var}_\alpha(A) \qquad (A \in \mathfrak{S}) \tag{1.4}$$

where

$$\alpha(T, B) = \sup_{|t| \leqq T} |1 - f(t, B)| \qquad (B \in \mathfrak{S}) \tag{1.5}$$

and T is an arbitrary but fixed positive number. In order to avoid superfluous complications in the formulae of the present paper we extend the definition of $W(T, A)$ $(A \in \mathfrak{S})$ by writing

$$W(-T, A) = W(T, A),$$

$W(0, A) = 0$. Thus for a fixed A $W(I, A)$ is an even function defined on the whole real axis. It is shown in [18] (cf the proof of Theorem 3. 2) that the set function (1. 4) is a finite (consequently also bounded) measure on the σ-ring $\mathfrak{S}$. In the sequel this measure will have a fundamental role.

The first theorem refers to this and to a closely related set function.

THEOREM 1. 1. *For every set $A \in \mathfrak{S}$*

$$\lim_{T \to 0} W(T, A) = 0, \tag{1.6}$$

i. e. $W(T, A)$ is continuous at $T = 0$ ($W(0, A)$ is obviously equal to 0). It is true furthermore that for every set $A \in \mathfrak{S}$ the set function

$$\mathrm{Var}_{\mu(a)}(A) \qquad (\mu(a, B) = \mathbf{P}(|\xi(B)| > a),\ B \in \mathfrak{S}) \tag{1.7}$$

depending on the number a is continuous at $a = \infty$, i. e.

$$\lim_{a \to \infty} \mathrm{Var}_{\mu(a)}(A) = 0. \tag{1.8}$$

PROOF. Let $B_1, \ldots, B_r$ be a system of disjoint sets of the σ-algebra $A\mathfrak{S}$. According to Theorem 3. 4 of [18] for every $\varepsilon > 0$ there can be found a $\delta > 0$ such that

$$-\log |f(t, C)| \leqq \varepsilon, \quad |\arg f(t, C)| \leqq \varepsilon,\text{[1]} \tag{1.9}$$

provided that $C \in \mathfrak{S}$ and $|t| \leqq \delta$. Let us divide the system $B_1, \ldots, B_r$ into two groups accordingly as $\arg f(t, B_k) > 0$ or $\arg f(t, B_k) \leqq 0$. By (1. 9) we have (if $\varepsilon < \frac{2\pi}{3}$).

$$\begin{gathered}\sum_k{}' \arg f(t, B_k) = \arg f(t, B') \leqq \varepsilon, \\ -\sum_k{}'' \arg f(t, B_k) = -\arg f(t, B'') \leqq \varepsilon,\end{gathered} \tag{1.10}$$

[1] Under $\arg z$ we understand the main value of this function: $-\pi < \arg z \leqq \pi$.

where the summations $\sum'$ and $\sum''$ refer to the subscripts of the sets belonging to the first and second groups, the sets B' and B'' denote the unions of the corresponding sets, respectively. (1. 10) implies that

$$\sum_{k=1}^{r} |\arg f(t, B_k)| \leqq 2\varepsilon \tag{1.11}$$

if $|t| \leqq \delta$. On the other hand, by (1. 9) we have also

$$-\log |f(t, B)| \leqq \varepsilon \qquad (B = B' + B'') \tag{1.12}$$

if $|t| \leqq \delta$. Using (1. 11) and (1. 12) we get

$$\begin{aligned} \sum_{k=1}^{r} |1-f(t, B_k)| &= \sum_{k=1}^{r} \left|1-e^{i \arg f(t, B_k)} |f(t, B_k)|\right| \leqq \\ &\leqq \sum_{k=1}^{r} \left|1-e^{i \arg f(t, B_k)}\right| + \sum_{k=1}^{r} (1-|f(t, B_k)|) \leqq \\ &\leqq \sum_{k=1}^{r} |\arg f(t, B_k)| - \sum_{k=1}^{r} \log |f(t, B_k)| \leqq 2\varepsilon - \log |f(t, B)| \leqq 3\varepsilon. \end{aligned} \tag{1.13}$$

Applying the inequality

$$\frac{1}{2a} \int_{-a}^{a} |1-f(t)|\, dt \geqq \frac{1}{10} \int_{|x| > \frac{1}{a}} dF(x), \tag{1.14}$$

where $F(x)$ is an arbitrary distribution function, $f(t)$ is the corresponding characteristic function and a is a positive number, we conclude

$$\sum_{k=1}^{r} \mathbf{P}\left(|\xi(B_k)| > \frac{1}{a}\right) \leqq \frac{10}{2a} \sum_{k=1}^{r} \int_{-a}^{a} |1-f(t, B_k)|\, dt \leqq 30\varepsilon \tag{1.15}$$

whenever $\frac{1}{a} \geqq \frac{1}{\delta}$. Thus the second assertion of our theorem is proved.

Our first assertion follows from the inequality

$$\begin{aligned} |1-f(t, B_k)| &= \left| \int_{-\infty}^{\infty} (1-e^{itx})\, dF(x, B_k) \right| \leqq \\ &\leqq |t| \left| \int_{|x| \leqq a} x\, dF(x, B_k) \right| + \frac{t^2}{2} \int_{|x| \leqq a} x^2\, dF(x, B_k) + 2\mathbf{P}(|\xi(B_k)| > a) \end{aligned} \tag{1.16}$$

where a is a positive number and from Theorem 3. 8 of [18]. This completes the proof.

The following theorem will be needed first of all in the proof of the existence of the integral (1. 3).

THEOREM 1.2. *Let $C_k^{(n)}$ $(k, n=1,2,\ldots)$ be a system of sets of the σ-ring $\mathfrak{S}$ for which $C_i^{(n)}C_k^{(n)}=0$ if $i\neq k$ $(n=1,2,\ldots)$ and $\delta_k^{(n)}$ $(k, n=1,2,\ldots)$ a double sequence of real numbers for which $\lim_{n\to\infty}\sup_k|\delta_k^{(n)}|=0$. In this case*

(1.17)
$$\sum_k \delta_k^{(n)}\xi(C_k^{(n)})\Rightarrow 0$$
if $n\to\infty$.

PROOF. For a fixed n the series (1.17) converges with probability 1 regardless of the order of summation. In fact, this holds for the series
$$\sum_k \xi(C_k^{(n)})$$
and the set of numbers $\{\delta_k^{(n)}\}$ is bounded, hence an easy argument referring to the three series theorem of KOLMOGOROV shows the truth of the preceding assertion.

Using an inequality of the type (1.16) we obtain

$$|1-\prod_k f(t\delta_k^{(n)}, C_k^{(n)})|\leqq\sum_k|1-f(t\delta_k^{(n)}, C_k^{(n)})|\leqq$$
$$\leqq|t|\sum_k|\delta_k^{(n)}|\left|\int_{|x|\leqq a} x\,dF(x, C_k^{(n)})\right|+ \tag{1.18}$$
$$+\frac{t^2}{2}\sum_k(\delta_k^{(n)})^2\int_{|x|\leqq a} x^2 dF(x, C_k^{(n)})+2\mathbf{P}(|\xi(C_k^{(n)})|>a)$$

where a is a positive number. By Theorem (3.8) of [18] there is a number $K_a>0$ such that
$$\sum_k\left|\int_{|x|\leqq a} x\,dF(x, C_k^{(n)})\right|\leqq K_a,$$
$$\sum_k\int_{|x|\leqq a} x^2 dF(x, C_k^{(n)})\leqq K_a.$$
We may suppose that $K_a\geqq 1$. First, using Theorem 1.1, we choose a so large that the third member on the right-hand side of (1.18) is smaller than ε (we suppose that $\varepsilon\leqq 1$). Next we choose n so that $|\delta_k^{(n)}|\leqq\frac{\varepsilon}{K_a}\leqq 1$ $(k=1,2,\ldots)$. In this case the sum on the right-hand side of (1.18) is at most $\varepsilon(|t|+t^2/2+1)$. Thus our theorem is proved.

§ 3. The existence of the integral

Now we return to our original problem, the discussion of the existence of the integral defined in § 1. The main theorem regarding this matter is the following:

THEOREM 1.3. *Let $\varphi(h)$ be a real function measurable with regard to the σ-ring $\mathfrak{S}$ and $A \in \mathfrak{S}$ a set. Suppose that there exists a $\delta > 0$ such that the series* (1.1) *converges with probability* 1 *regardless of the order of summation whenever the dividing point sequence $\{y_k\}$ has the property* $\sup_k (y_{k+1} - y_k) \leq \delta$. *In this case the integral* (1.3) *exists. In particular, every measurable bounded function is integrable.*

PROOF. Let $\{y_k^{(n)}\}$ be an infinitely fine dividing point double sequence. Let us unite the sequences $\{y_k^{(n)}\}$, $\{y_k^{(m)}\}$ and denote the new dividing point sequence by $\{z_k\}$.[2] If we introduce the notation

$$L_j = \{h : z_j \leq \varphi(h) < z_{j+1}\},$$

then

$$\sum_k y_k^{(n)} \xi(AH_k^{(n)}) - \sum_k y_k^{(m)} \xi(AH_k^{(m)}) =$$

$$(1.19) \qquad = \sum_k \sum_{j: L_j \subseteq H_k^{(n)}} (y_k^{(n)} - z_j) \xi(AL_j) - \sum_k \sum_{j: L_j \subseteq H_k^{(m)}} (y_k^{(m)} - z_j) \xi(AL_j).$$

In the sums on the right-hand side the set of the numbers $y_k^{(n)} - z_j$, $y_k^{(m)} - z_j$ is bounded and if $\delta_{k,j}^{(n)} = y_k^{(n)} - z_j$, then the properties of the double sequence $\{y_k^{(n)}\}$ imply that

$$\lim_{n \to \infty} \sup_k \max_{j: L_j \subseteq H_k^{(n)}} |\delta_{kj}^{(n)}| = 0.$$

Applying Theorem 1.2 it follows that the sequence of random variables in (1.19) tends stochastically to 0 as $n, m \to \infty$.

Thus the Cauchy's convergence criterion holds, hence there is a random variable to which the first member on the left-hand side of (1.19) converges stochastically. As it can be seen by a well-known argument, this limit is independent of the special choice of the double sequence $\{y_k^{(n)}\}$. Hence our theorem is proved.

REMARK. It is not difficult to see that if the series (1.1) converges for sufficiently fine dividing point sequences, then the same holds for any dividing point sequence. This and other statements occurring in our discussion follow from the fact that if the series of independent random variables

$$\sum_{k=1}^{\infty} \xi_k$$

converges with probability 1 regardless of the order of summation, then the

[2] For the sake of brevity we do not indicate that it depends on n and m.

same holds for

$$\sum_{k=1}^{\infty} c_k \xi_k$$

where c_k is a bounded sequence of real numbers.[3]

Now we deduce an inequality playing an important role in the proofs of the present paper.

THEOREM 1.4. *Let $\varphi(h)$ be a measurable function for which $|\varphi(h)| \leq K < \infty$. Then, denoting by $g(t, A)$ the characteristic function of the integral of $\varphi(h)$ over the set A ($A \in \mathfrak{S}$), we have*

$$|1-g(t, A)| \leq W(tK, A).$$

PROOF. Let $\{y_k^{(n)}\}$ be an infinitely fine dividing point double sequence and $\{H_k^{(n)}\}$ the corresponding sequence of subdivisions determined by $\varphi(h)$. If $g_n(t, A)$ denotes the characteristic function of the random variable

$$\sum_k y_k^{(n)} \xi(AH_k^{(n)}),$$

then

$$|1-g_n(t, A)| \leq \sum_k |1-f(ty_k^{(n)}, AH_k^{(n)})| \leq$$

$$\leq \sum_k W(ty_k^{(n)}, AH_k^{(n)}) \leq \sum_k W(tK, AH_k^{(n)}) = W(tK, A).$$

Taking the limit $n \to \infty$, we obtain our assertion. Q. e. d.

§ 4. A necessary condition for the existence of the integral (1.3)

In this section we prove the following

THEOREM 1.5. *Let $\varphi(h)$ be a measurable function integrable over the set $A \in \mathfrak{S}$ and a_n, b_n ($n = 1, 2, \ldots$) a pair of sequences with the properties $a_{n+1} \leqq a_n$, $b_{n+1} \geqq b_n$ ($n = 1, 2, \ldots$), $\lim\limits_{n\to\infty} a_n = -\infty$, $\lim\limits_{n\to\infty} b_n = \infty$. In this case the sequence of random variables*

(1.20) $$\eta_n = \int_{A_n} \varphi(h)\, \xi(dA) \qquad (A_n = \{h : h \in A, a_n \leqq \varphi(h) < b_n\})$$

converges with probability 1 and

(1.21) $$\int_A \varphi(h)\, \xi(dA) = \lim_{n\to\infty} \eta_n.$$

PROOF. We may suppose that $a_n < 0$, $b_n > 0$,

$$\sup_n (b_{n+1} - b_n) < \infty, \quad \sup_n (a_n - a_{n+1}) < \infty \qquad (n = 1, 2, \ldots).$$

[3] This statement is a simple consequence of the three series theorem of KOLMOGOROV.

Let us unite these sequences into one dividing point sequence $\{y_n\}$ and consider the series

$$\sum_n \int_{AH_n} (y_n - \varphi(h))\,\xi(dA) \tag{1.22}$$

where $\{H_n\}$ is the subdivision determined by the sequence $\{y_n\}$ and the function $\varphi(h)$. We do not know yet that the series (1. 22) converges. Denoting by $f_n(t)$ the characteristic function of the n-th member of the series (1. 22), it follows from Theorem 1. 4 that

$$|1-f_n(t)| \leq W(\delta t, AH_n) \tag{1.23}$$

where $\delta = \sup\limits_n (y_{n+1}-y_n)$. (1. 23) implies

$$\sum_n |1-f_n(t)| \leq W(\delta t, A). \tag{1.24}$$

As the terms of the series (1. 22) are evidently independent, this series converges with probability 1 regardless of the order of summation (cf. [5], p. 115, Theorem 2. 7). The sum (1. 22) equals

$$\sum_n y_n \xi(AH_n) - \sum_n \int_{AH_n} \varphi(h)\,\xi(dA),^4$$

hence, taking Theorem 1. 3 into account, it follows that if $\varphi(h)$ is integrable over A, then the sequences of the type (1. 20) converge with probability 1.

Now we are going to verify the relation (1. 21). Let η denote the limiting variable of the sequence η_n, and choose an infinitely fine dividing point double sequence $\{y_n^{(N)}\}$ with the property $\{y_n^{(N)}\} \subset \{y_n^{(N+1)}\}$ and $\{y_n^{(1)}\} = \{y_n\}$. If $\{H_n^{(N)}\}$ is the corresponding sequence of subdivisions determined by $\varphi(h)$, then, using the precedings, we have

$$\sum_n \int_{AH_n^{(N)}} \varphi(h)\,\xi(dA) = \eta \tag{1.25}$$

for every N. Let $f_n^{(N)}(t)$ denote the characteristic function of the n-th term of the series

$$\sum_n \int_{AH_n^{(N)}} (y_n^{(N)} - \varphi(h))\,\xi(dA). \tag{1.26}$$

If $f^{(N)}(t)$ denotes the characteristic function of the sum (1. 26), then by Theorem 1. 4 we have

$$|1-f^{(N)}(t)| \leq \sum_n |1-f_n^{(N)}(t)| \leq W(\delta^{(N)} t, A) \tag{1.27}$$

[4] We have used here a trivial special case of Theorem 2. 4.

where $\delta^{(N)} = \sup_n (y_{n+1}^{(N)} - y_n^{(N)})$. By (1. 27) and Theorem 1. 1

$$f^{(N)}(t) \Longrightarrow 1 \quad \text{if} \quad N \to \infty.$$

(1. 25) implies that the sequence (1. 26) (which depends on N) converges stochastically to the random variable

$$\int_A \varphi(h)\,\xi(dA) - \eta.$$

Hence this is equal to 0 with probability 1. Q. e. d.

II. THE PROPERTIES OF THE INTEGRAL $\int_A \varphi(h)\,\xi(dA)$

§ 1. Elementary properties of the integral

It is only for the sake of the systematic treatment that we mention the following almost trivial theorems:

THEOREM 2. 1. *If $\varphi(h)$ is integrable over the set $A \in \mathfrak{S}$, then the same holds also for $c\varphi(h)$ where c is a real constant and*

$$\int_A c\varphi(h)\,\xi(dA) = c\int_A \varphi(h)\,\xi(dA). \tag{2. 1}$$

THEOREM 2. 2. *If $\varphi(h)$ is integrable over the sets $A_1 \in \mathfrak{S}$, $A_2 \in \mathfrak{S}$ where $A_1 A_2 = 0$, then it is integrable also over $A_1 + A_2$ and*

$$\int_{A_1+A_2} \varphi(h)\,\xi(dA) = \int_{A_1} \varphi(h)\,\xi(dA) + \int_{A_2} \varphi(h)\,\xi(dA). \tag{2. 2}$$

The following theorem requires a little more complicated argument than the corresponding theorem for Lebesgue integrals.

THEOREM 2. 3. *If $\varphi(h)$ is a measurable and bounded step function over the set $A \in \mathfrak{S}$, i. e. there are disjoint sets $A_1, A_2, \ldots$ $\left(\sum_{k=1}^{\infty} A_k = A\right)$ of $\mathfrak{S}$ such that $\varphi(h) = \varphi_k$ if $h \in A_k$ and $|\varphi_k| \leq K$ $(k = 1, 2, \ldots)$ where K is a constant, then*

$$\int_A \varphi(h)\,\xi(dA) = \sum_{k=1}^{\infty} \varphi_k\,\xi(A_k). \tag{2. 3}$$

A similar but more general assertion follows from Theorem 2. 4 for u-integrable functions taking on a countable number of different values.

§ 2. The complete additiveness of the indefinite integral

In this § our aim is to prove the following

THEOREM 2.4. *If the function* $\varphi(h)$ *is integrable over all sets* $A \in \mathfrak{S}$, *then*

$$\eta(A) = \int_A \varphi(h)\, \xi(dA) \tag{2.4}$$

is a completely additive stochastic set function.[5]

PROOF. It is quite easy to argue that $\eta(A)$ is an additive set function. In fact, if the sets $A_k \in \mathfrak{S}$ $(k=1, 2, \ldots, r)$ are disjoint, then the random variables $\eta(A_k)$ $(k=1, 2, \ldots, r)$ must be independent and Theorem 2.2 ensures the fulfilment of the remaining part of this assertion.

Now we prove that the set function η is also completely additive. First we consider the case of a bounded function and suppose that $|\varphi(h)| \leq K$. According to Theorem 2.1 of [18] we have to verify the following criterion: if $B_1, B_2, \ldots$ is a non-increasing sequence of sets of $\mathfrak{S}$ such that $\lim_{n\to\infty} B_n = 0$, then

$$\eta(B_n) \Rightarrow 0 \quad \text{if} \quad n \to \infty. \tag{2.5}$$

Denoting by $f_n(t)$ the characteristic function of $\eta(B_n)$, Theorem 1.4 implies

$$|1-f_n(t)| \leqq W(Kt, B_n). \tag{2.6}$$

As for every T $W(T, A)$ is a bounded measure on the σ-ring $\mathfrak{S}$, it follows that

$$\lim_{n\to\infty} W(Kt, B_n) = 0. \tag{2.7}$$

Our assertion follows from (2.6) and (2.7).

Now we consider the general case. Let us introduce the notation $\mathfrak{S}_N = D_N \mathfrak{S}$ where $D_N = \{h: -N \leq \varphi(h) \leq N\}$. We shall denote by $\mathfrak{R}$ the ring of those sets A which are of the form

$$A = A_{i_1} + \cdots + A_{i_r} \quad \text{where} \quad A_{i_k} \in \mathfrak{S}_{i_k} \qquad (k=1, \ldots, r).$$

Our argument for the case of a bounded function shows that the set function η is completely additive on $\mathfrak{R}$. In fact, if $A_1, A_2, \ldots$ is a sequence of disjoint sets of $\mathfrak{R}$ for which $A = \sum_{k=1}^{\infty} A_k \in \mathfrak{R}$, then for some N we have $A \in \mathfrak{S}_N$. But since $A_k \subset A$ $(k=1, 2, \ldots)$, we have also $A_k \in \mathfrak{S}_N$ $(k=1, 2, \ldots)$. As $\varphi(h)$ is bounded on the σ-ring $\mathfrak{S}_N$, it follows that

$$\eta(A) = \sum_{k=1}^{\infty} \eta(A_k).$$

[5] The definition of this notion is given in [18], p. 216.

The following step is to prove that the set function η can be extended to the smallest σ-ring $\mathfrak{S}(\mathfrak{R})$ containing $\mathfrak{R}$ (cf. [18], p. 233). Obviously $\mathfrak{S}(\mathfrak{R}) = \mathfrak{S}$. Let $B_1, B_2, \ldots$ be an arbitrary sequence of disjoint sets of $\mathfrak{R}$. We shall show that the series

(2. 8)
$$\sum_{k=1}^{\infty} \eta(B_k)$$

converges with probability 1 regardless of the order of summation. By Theorem 3. 2 of [18] this property ensures the possibility of the extension. Let $C_{N_r k_s^{(r)}} = (D_{N_r+1} - D_{N_r}) B_{k_s^{(r)}}$ $(N_r < N_{r+1}, k_s^{(r)} < k_{s+1}^{(r)}; s = 1, 2, \ldots; r = 1, 2, \ldots)$.

Since $|\varphi(h)| \leq N_r + 1$ for $h \in C_{N_r} = \sum_{s=1}^{\infty} C_{N_r k_s^{(r)}}$, it follows that

(2. 9)
$$\eta(C_{N_r}) = \sum_{s=1}^{\infty} \eta(C_{N_r k_s^{(r)}})$$

and this series converges with probability 1 regardless of the order of summation. On the other hand, $\varphi(h)$ is integrable over the set $C = \sum_{r=1}^{\infty} C_{N_r}$, hence by Theorem 1. 5 (choosing the sequences $a_r = -N_r - 1$, $b_r = N_r + 1$)

(2. 10)
$$\eta(C) = \sum_{r=1}^{\infty} \eta(C_{N_r}).$$

(This series must converge also if we omit an arbitrary set of terms, hence it converges with probability 1 regardless of the order of summation.[6]) Comparing (2. 9) and (2. 10), it follows that the series

(2. 11)
$$\sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \eta(C_{N_r k_s^{(r)}})$$

converges with probability 1. Now we prove a

LEMMA. *Let ξ_{ik} $(i, k = 1, 2, \ldots)$ be independent random variables. Suppose that for every system of sequences $i_r, k_s^{(r)}$ $(s = 1, 2, \ldots; r = 1, 2, \ldots)$ with $i_r < i_{r+1}$, $k_s^{(r)} < k_{s+1}^{(r)}$ $(s = 1, 2, \ldots; r = 1, 2, \ldots)$ the series*

(2. 12)
$$\sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \xi_{i_r k_s^{(r)}}$$

converges with probability 1. In this case the series

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \xi_{ik}$$

converges with probability 1 regardless of the order of summation.

[6] This can be seen e. g. by the aid of the three series theorem of KOLMOGOROV.

PROOF. By condition there is a $\delta > 0$ such that for every t for which $|t| \leqq \delta$, the infinite double product

$$\prod_{r=1}^{\infty} \prod_{s=1}^{\infty} f_{i_r k_s^{(r)}}(t) \tag{2. 13}$$

converges where $f_{ik}(t)$ is the characteristic function of the random variable ξ_{ik}.

Let us fix the number t ($|t| \leqq \delta$) and introduce the notation

$$f_{ik}(t) = \varrho_{ik} e^{i\alpha_{ik}} \qquad (i, k = 1, 2, \ldots)$$

where $-\pi < \alpha_{ik} \leqq \pi$. Since (2. 13) converges, it follows that

$$\prod_{r=1}^{\infty} \prod_{s=1}^{\infty} \varrho_{i_r k_s^{(r)}}$$

does also. Hence a well-known argument shows that

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (1-\varrho_{ik}) < \infty .$$

Now consider the double series

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \alpha_{ik}$$

and prove that it converges absolutely. Let us suppose that it does not hold. In this case either the sum of the positive terms or that of the negative terms diverges. We consider the first case, the second one can be treated similarly. Then there is a countable number of rows of the matrix (α_{ik}) which contain the positive elements. If $j_r, l_s^{(r)}$ ($s = 1, 2, \ldots$; $r = 1, 2, \ldots$) denote the positions of the positive elements, then the convergence of the infinite product

$$\prod_{s} f_{j_r l_s^{(r)}} \qquad (r = 1, 2, \ldots)$$

implies that

$$\alpha_{j_r} = \sum_{s} \alpha_{j_r l_s^{(r)}} < \infty .$$

We can distinguish two cases accordingly as $\alpha_{j_r} \leqq \pi$ for every large r or $\alpha_{j_r} > \pi$ for an infinite number of the r's. In the first case the convergence of the infinite product

$$\prod_{r=1}^{\infty} \prod_{s} f_{j_r l_s^{(r)}}$$

implies

$$\sum_{r=1}^{\infty} \alpha_{j_r} < \infty$$

what is a contradiction. Considering the second case, let $i_1, i_2, \ldots$ denote a

subsequence of the sequence $j_1, j_2, \ldots$ for which $\alpha_{i_r} \geqq \pi$ $(r = 1, 2, \ldots)$. Then in the i_r-th row of the matrix there is a finite or infinite number of positions, whose subscripts will be denoted by $k_1^{(r)}, k_2^{(r)}, \ldots$, such that

$$\frac{\pi}{2} \leqq \sum_s \alpha_{i_r k_s^{(r)}} \leqq \pi.$$

This is, however, also a contradiction, since the infinite product

$$\prod_{i=1}^{\infty} \prod_{k} e^{i\alpha_{ik}}$$

converges. Thus

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} |\alpha_{ik}| < \infty.$$

In view of the inequality

$$|1-f_{ik}(t)| \leqq 1-\varrho_{ik}+|\alpha_{ik}|$$

we obtain

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} |1-f_{ik}(t)| < \infty \quad \text{if} \quad |t| \leqq \delta,$$

hence our Lemma follows (cf. [5], p. 115, Theorem 2.7 and p. 118, Corollary 1).

Applying the Lemma for the random variables $\xi_{ik} = \eta(C_{ik})$, we conclude that the series

(2.14) $$\sum_{k=1}^{\infty} \sum_{N=1}^{\infty} \eta(C_{Nk})$$

converges with probability 1 regardles of the order of summation. By Theorem 1.5 we have

(2.15) $$\eta(B_k) = \sum_{N=0}^{\infty} \eta(C_{Nk}),$$

thus the series (2.8) converges with probability 1 regardless of the order of summation.

Let $\eta^*(A)$ $(A \in \mathfrak{S})$ denote the extended set function of the set function $\eta(A)$ $(A \in \mathfrak{R})$. If E_N is the following set:

$$E_N = \{h : h \in B, \ -N \leqq \varphi(h) < N\} \quad \text{where} \quad B \in \mathfrak{S},$$

then by Theorem 1.5

(2.16) $$\lim_{N\to\infty} \int_{E_N} \varphi(h)\,\xi(dA) = \int_B \varphi(h)\,\xi(dA).$$

Now, since η^* and η coincide on $\mathfrak{R}$ and η is the indefinite integral (2.4), we have

(2.17) $$\eta^*(B) = \lim_{N\to\infty} \eta^*(E_N) = \lim_{N\to\infty} \eta(E_N) = \lim_{N\to\infty} \int_{E_N} \varphi(h)\,\xi(dA).$$

On the basis of (2. 16) and (2. 17) we may write

$$\nu^*(B) = \int_B \varphi(h)\,\xi(dA).$$

Since the extension process leads to a completely additive stochastic set function, this property of ν^* implies the assertion of our theorem.

REMARK. *Let $\varphi(h)$ be a measurable function taking on a countable number of different values on the set $A \in \mathfrak{S}$, i. e. there are measurable disjoint sets $A_1, A_2, \ldots$ of the σ-ring $A\mathfrak{S}$ such that $A = \sum_{k=1}^{\infty} A_k$ and $\varphi(h) = \varphi_k$ if $h \in A_k$ $(k = 1, 2, \ldots)$. If $\varphi(h)$ is integrable over the sets of the type $A_{i_1} + A_{i_2} + \cdots$ where $i_1, i_2, \ldots$ is an arbitrary sequence of natural numbers, then (it is integrable over A and)*

$$\int_A \varphi(h)\,\xi(dA) = \sum_{k=1}^{\infty} \varphi_k\,\xi(A_k)$$

where the series on the right-hand side converges with probability 1 *regardless of the order of summation.*

PROOF. Consider the σ-algebra of the sets $A_{i_1} + A_{i_2} + \cdots$ and denote it by $\mathfrak{S}$. By condition $\varphi(h)$ is u-integrable over the set A with respect to $\mathfrak{S}$ and the completely additive set function ξ. Since $\varphi(h)$ is integrable over $A = A_1 + A_2 + \cdots$ with respect to $\mathfrak{S}$ and ξ, it is integrable with respect to $\mathfrak{S}$ and ξ too. Applying Theorem 2. 4 for $\bar{\mathfrak{S}}$ instead of $\mathfrak{S}$, the assertion follows.

§ 3. Further properties of the integral

Two properties of the classical Lebesgue—Radon integral cannot be formulated in case of the integral introduced by Definition 3. They will, however, hold in case of the u-integral. The first of these properties is that if $\varphi(h)$ and $\psi(h)$ are two measurable functions such that $|\varphi(h)| \leq \psi(h)$, moreover $\psi(h)$ is integrable with respect to some finite measure over a measurable set A, then the same holds for $\varphi(h)$.

An example for this is the following: Let H be the set of the natural numbers and $\mathfrak{S}$ the σ-algebra of all subsets of H. We define the set function $\xi(A)$ as follows:

$$\xi(A) = \sum_{h \in A} (-1)^{h+1} \frac{1}{h^2}. \tag{2. 18}$$

It is easy to see that the function

$$\left.\begin{aligned} \psi(2h-1) &= 2h, \\ \psi(2h) &= 2h \end{aligned}\right\} \quad (h = 1, 2, \ldots) \tag{2. 19}$$

is integrable over H, but the same is not true for

(2. 20) $$\varphi(h)=h \qquad (h=1,2,\ldots)$$

though $0<\varphi(h)\leqq\psi(h)$ for every h.

A similar example can be given for the assertion that the integrability of two functions $\varphi_1(h)$ and $\varphi_2(h)$ over a set $A\in\mathfrak{S}$ does not imply the integrability of $\varphi_1(h)+\varphi_2(h)$ over the same set.

Let H and $\mathfrak{S}$ be the same as before and define the functions $\varphi_1(h)$, $\varphi_2(h)$ as follows:

(2. 21) $$\left.\begin{aligned}\varphi_1(2h)&=(-1)^h 2h,\\ \varphi_1(2h-1)&=(-1)^h 2h\end{aligned}\right\}\quad (h=1,2,\ldots),$$

(2. 22) $$\left.\begin{aligned}\varphi_2(1)&=0,\\ \varphi_2(2h)&=(-1)^{h+1}2h,\\ \varphi_2(2h+1)&=(-1)^{h+1}2h\end{aligned}\right\}\quad (h=1,2,\ldots).$$

In this case the function $\varphi_1(h)+\varphi_2(h)$ vanishes on the set of the even numbers and its integral does not exist on the set of the positive odd numbers, since the series defining this integral is not absolutely convergent.

As it will be shown in the sequel, these anomalies do not occur in case of the u-integral. We prove two theorems, the first of which refers to the u-integrability of the sum of two u-integrable functions, while the second one, referring to the existence of the u-integral of a majorated function (Theorem 2. 8), requires further preparations.

THEOREM 2. 5. *If the functions $\varphi_1(h)$ and $\varphi_2(h)$ are u-integrable over the set $A\in\mathfrak{S}$, then the same holds for $\varphi_1(h)+\varphi_2(h)$ and*

(2. 23) $$\int\limits_A^{\cdot}(\varphi_1(h)+\varphi_2(h))\xi(dA)=\int\limits_A^{\cdot}\varphi_1(h)\xi(dA)+\int\limits_A^{\cdot}\varphi_2(h)\xi(dA).$$

PROOF. Let $\{y_k^{(n)}\}$ be an infinitely fine dividing point double sequence and $\{H_k^{(n)}\}$ the corresponding sequence of subdivisions determined by the function $\varphi_1(h)+\varphi_2(h)$. Now consider the following series:

(2. 24) $$\begin{aligned}&\sum_{k=-\infty}^{\infty}\Big[y_k^{(n)}\xi(AH_k^{(n)})-\int\limits_{AH_k^{(n)}}^{\cdot}\varphi_1(h)\xi(dA)-\int\limits_{AH_k^{(n)}}^{\cdot}\varphi_2(h)\xi(dA)\Big]=\\ &=\sum_{k=-\infty}^{\infty}y_k^{(n)}\xi(AH_k^{(n)})-\sum_{k=-\infty}^{\infty}\int\limits_{AH_k^{(n)}}^{\cdot}\varphi_1(h)\xi(dA)-\sum_{k=-\infty}^{\infty}\int\limits_{AH_k^{(n)}}^{\cdot}\varphi_2(h)\xi(dA).\end{aligned}$$

The second and third series on the right-hand side converge with probability 1 regardless of the order of summation and their sum equals the integrals over

A of $\varphi_1(h)$ and $\varphi_2(h)$, resp. We do not know yet anything about the convergence of the series on the left-hand side.

Let $\{z_s^{(m)}\}$ be an infinitely fine dividing point double sequence, $\{L_s^{(m)}\}$ and $\{M_s^{(m)}\}$ the corresponding sequences of subdivisions determined by the functions $\varphi_1(h)$ and $\varphi_2(h)$, resp. We introduce the notation $f_k^{(n)}(t)$ for the characteristic function of the k-th member on the left-hand side of (2. 24). Since

$$y_k^{(n)}\xi(AH_k^{(n)})-\int\limits_{AH_k^{(n)}}\varphi_1(h)\xi(dA)-\int\limits_{AH_k^{(n)}}\varphi_2(h)\xi(dA)=$$

$$(2.\ 25)\qquad =\lim_{m\to\infty}\operatorname{st}\{y_k^{(n)}\xi(AH_k^{(n)})-\sum_s z_s^{(m)}\xi(AH_k^{(n)}L_s^{(m)})-\sum_r z_r^{(m)}\xi(AH_k^{(n)}M_r^{(m)})\}=$$

$$=\lim_{m\to\infty}\operatorname{st}\sum_{s,r}(y_k^{(n)}-z_s^{(m)}-z_r^{(m)})\xi(AH_k^{(n)}L_s^{(m)}M_r^{(m)})$$

and in the last member $|y_k^{(n)}-z_s^{(m)}-z_r^{(m)}|\leq 2\delta_n$ for large m's where $\delta_n=$ $=\sup\limits_k(y_{k+1}^{(n)}-y_k^{(n)})$, it follows that

$$|1-f_k^{(n)}(t)|=|1-\lim_{m\to\infty}\prod_{s,r}f(t(y_k^{(n)}-z_s^{(m)}-z_r^{(m)}),AH_k^{(n)}L_s^{(m)}M_r^{(m)})|\leq$$

$$\leq\lim_{m\to\infty}\sum_{s,r}|1-f(t(y_k^{(n)}-z_s^{(m)}-z_r^{(m)}),AH_k^{(n)}L_s^{(m)}M_r^{(m)})|\leq W(2\delta_n t,AH_k^{(n)}).$$

Hence, by summation, we get

$$(2.\ 26)\qquad \sum_{k=-\infty}^{\infty}|1-f_k^{(n)}(t)|\leq W(2\delta_n t,A).$$

This shows that the series on the left-hand side, consequently also the first series on the right-hand side of (2. 24) converges with probability 1 regardless of the order of summation. Taking the limit $n\to\infty$ in (2. 26) and using Theorem 1. 1, it follows that the right-hand side of (2. 24) tends stochastically to 0 which implies (2. 23).

We can repeat the argument for an arbitrary measurable subset of the set A and thus the assertion follows.

§ 4. Two auxiliary theorems

THEOREM 2. 6. *A non-negative measurable function $\varphi(h)$ is u-integrable over the set $A\in\mathfrak{S}$ if and only if for every dividing point sequence $\{y_k\}$, for which $y_0=0$, the relation*

$$(2.\ 27)\qquad \sum_{k=0}^{\infty}W(y_k,AH_k)<\infty\qquad(t\geq 0)$$

holds where $\{H_k\}$ is the subdivision corresponding to $\varphi(h)$ and $\{y_k\}$.

PROOF. The sufficiency of the condition is a simple consequence of Theorem 1.3. In fact, if $B \in A\mathfrak{S}$, then (2.27) implies that for every t

$$\sum_{k=0}^{\infty} W(ty_k, BH_k) \leqq \sum_{k=0}^{\infty} W(ty_k, AH_k) < \infty,$$

hence

$$\sum_{k=0}^{\infty} \sup_{|z| \leqq t} |1 - f(zy_k, BH_k)| < \infty$$

which implies the convergence in every order of the series

$$\sum_{k=0}^{\infty} y_k \xi(BH_k).$$

The other part of the theorem is less obvious. We prove it in an indirect way and suppose that there exist disjoint sets B_{kl} $(l = 1, \ldots, l_k;$ $k = 0, 1, 2, \ldots)$ such that $B_{kl} \in AH_k\mathfrak{S}$ $(l = 1, \ldots, l_k;\ k = 0, 1, 2, \ldots)$ and

$$\sum_{k=0}^{\infty} \sum_{l=1}^{l_k} \sup_{|z| \leqq t} |1 - f(zy_k, AB_{kl})| = \infty. \tag{2.28}$$

In view of the inequality

$$|1 - g(t)| \leqq \frac{t^2}{2} \int\limits_{|x| \leqq \varepsilon} x^2 dG(x) + |t| \left| \int\limits_{|x| \leqq \varepsilon} x dG(x) \right| + 2 \int\limits_{|x| > \varepsilon} dG(x) \leqq$$

$$\leqq 2\left(t^2 + \frac{1}{t^2}\right)\left(\int\limits_{|x| \leqq \varepsilon} x^2 dG(x) + \left|\int\limits_{|x| \leqq \varepsilon} x dG(x)\right| + \int\limits_{|x| > \varepsilon} dG(x)\right),$$

valid for every t, every $\varepsilon > 0$ and every distribution function, the characteristic function of which is $g(t)$, we may write

$$\sum_{k=0}^{\infty} \sum_{l=1}^{l_k} \left(\int\limits_{|x| \leqq \varepsilon} x^2 dF\left(\frac{x}{y_k}, AB_{kl}\right) + \left| \int\limits_{|x| \leqq \varepsilon} x dF\left(\frac{x}{y_k}, AB_{kl}\right)\right| + \int\limits_{|x| > \varepsilon} dF\left(\frac{x}{y_k}, AB_{kl}\right)\right) = \infty.$$

Hence the three series theorem of KOLMOGOROV implies that the series of independent random variables

$$\sum_{k=0}^{\infty} \sum_{l=1}^{l_k} y_k \xi(B_{kl})$$

does not converge with probability 1 regardless of the order of summation.

This leads to a contradiction. Let $K = \sup\limits_k (y_{k+1} - y_k)$ and denote by $f_{kl}(t)$ the characteristic function of the random variable

$$\int\limits_{B_{kl}} \varphi(h) \xi(dA) - y_k \xi(B_{kl}) = \int\limits_{B_{kl}} (\varphi(h) - y_k) \xi(dA). \tag{2.29}$$

By Theorem 1. 4

$$|1-f_{kl}(t)| \leq W(Kt, B_{kl}). \tag{2.30}$$

This implies that

$$\sum_{k=0}^{\infty} \sum_{l=1}^{l_k} |1-f_{kl}(t)| < \infty, \tag{2.31}$$

hence the series

$$\sum_{k=0}^{\infty} \sum_{l=1}^{l_k} \left(\int_{B_{kl}} \varphi(h)\, \xi(dA) - y_k\, \xi(B_{kl}) \right) \tag{2.32}$$

converges with probability 1 regardless of the order of summation. Since $\varphi(h)$ is integrable over all elements of the σ-ring $A\mathfrak{S}$, by Theorem 2. 4 the series

$$\sum_{k=0}^{\infty} \sum_{l=1}^{l_k} \int_{B_{kl}} \varphi(h)\, \xi(dA) \tag{2.33}$$

has also this property. Thus

$$\sum_{k=0}^{\infty} \sum_{l=1}^{l_k} y_k\, \xi(B_{kl}) \tag{2.34}$$

converges with probability 1 regardless of the order of summation what is a contradiction. Q. e. d.

THEOREM 2. 7. *Let $\varphi(h)$ and $\psi(h)$ be two non-negative measurable functions such that $\varphi(h) \leq \psi(h)$ and $\psi(h)$ is u-integrable over the set A $(A \in \mathfrak{S})$. If $\{y_k\}$ $(y_0 = 0)$ is a dividing point sequence, then we have*

$$\sum_{k=0}^{\infty} W(y_k, AL_k) \leq \sum_{k=0}^{\infty} W(y_k, AH_k) \tag{2.35}$$

where $\{L_k\}$ and $\{H_k\}$ are the subdivisions corresponding to the functions $\varphi(h)$ and $\psi(h)$, respectively.

PROOF. By definition

$$\left.\begin{aligned} L_k &= \{h : y_k \leq \varphi(h) < y_{k+1}\}, \\ H_k &= \{h : y_k \leq \psi(h) < y_{k+1}\} \end{aligned}\right\} \quad (k=0, 1, 2, \ldots),$$

hence

$$\sum_{k=0}^{j} H_k \subseteq \sum_{k=0}^{j} L_k \qquad (j=0, 1, 2, \ldots), \tag{2.36}$$

$$H_j L_k = 0 \quad \text{for} \quad k > j. \tag{2.37}$$

From these we conclude that

$$\sum_{j=0}^{\infty} W(y_j, AH_j) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} W(y_j, AH_jL_k) =$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{j} W(y_j, AH_jL_k) = \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} W(y_j, AH_jL_k) =$$

$$= \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} W(y_k, AH_jL_k) = \sum_{k=0}^{\infty} W\left(y_k, A\sum_{j=k}^{\infty} H_jL_k\right) = \sum_{k=0}^{\infty} W(y_k, AL_k)$$

what was to be proved.

§ 5. The theorem relative to the majorated function and the bounded convergence theorem

In § 3 of this Chapter we have seen that if a function has an integrable majorant, then this does not imply in general the integrability of the majorated function. For a positive assertion in this direction the same additional assumption is needed which has had a fundamental role in Theorem 2. 4. Our result is contained in

THEOREM 2. 8. *Let $\varphi(h)$ and $\psi(h)$ be measurable functions such that $|\varphi(h)| \leq \psi(h)$ and $\psi(h)$ is u-integrable over the set $A \in \mathfrak{S}$. Then $\varphi(h)$ is also u-integrable over the set A.*

PROOF. Let be $B \in A\mathfrak{S}$. If B_1 and B_2 denote those subsets of B where $\varphi(h) \geq 0$ and $\varphi(h) < 0$, resp., then Theorems 2. 6 and 2. 7 imply the integrability of $\varphi(h)$ over B_1 and B_2. Applying Theorem 2. 2, the assertion follows.

In accordance with the precedings, the analogy to the bounded convergence theorem contains also more assumptions than simple integrability. It seems, however, that Theorems 2. 7 and 2. 8 express unexpected good properties of our integral. We have only to remind of our paper [18] where an example is given for a completely additive stochastic set function which cannot be represented as the sum of positive and negative parts (cf. pp. 261—262). A remarkable situation is that though in case of the Radon integral the generalized measure can always be decomposed into a difference of two measures, our Theorems 2. 7 and 2. 8 need not more assumptions than the corresponding Radon integral theorems.

We shall not consider the integrability term by term of series of functions and other statements, but prove the following theorem from which these can easily be deduced:

THEOREM 2. 9. *Let $\varphi_N(h)$ $(N = 1, 2, \ldots)$ be a sequence of measurable functions such that $|\varphi_N(h)| \leq \psi(h)$ where $\psi(h)$ is u-integrable over the set*

$A \in \mathfrak{S}$. *Suppose furthermore that the limit* $\varphi(h) = \lim\limits_{N\to\infty} \varphi_N(h)$ *exists. In this case the functions* $\varphi(h), \varphi_N(h)$ $(N = 1, 2, \ldots)$ *are also u-integrable over the set* A *and we have the limit relation*

$$(2.38) \qquad \int\limits_A \varphi_N(h)\,\xi(dA) \Rightarrow \int\limits_A \varphi(h)\,\xi(dA) \quad \textit{if} \quad N \to \infty.$$

PROOF. The assertion regarding the integrability of the functions $\varphi(h), \varphi_N(h)$ $(N = 1, 2, \ldots)$ follows at once from Theorem 2. 8. According to Theorem 2. 5, we may restrict ourselves to the case when $\varphi(h) = 0, \varphi_N(h) \geq 0$ $(N = 1, 2, \ldots)$.

First we consider the case of a sequence majorated by a constant and suppose that $\varphi_N(h) \leq M$. Let A_N denote the following set:

$$A_N = \{h : \varphi_N(h) \leq \varrho\}$$

where ϱ is a positive constant. Denoting by $f_N(t)$ the characteristic function of the random variable

$$\int\limits_A \varphi_N(h)\,\xi(dA),$$

by Theorem 1. 4 we have

$$(2.39) \qquad |1 - f_N(t)| \leq W(\varrho t, A_N) + W(Mt, A - A_N) \leq W(\varrho t, A) + W(Mt, A - A_N).$$

Let t be a fixed number and choose first ϱ so small that the first term on the right-hand side of (2. 39) is less than $\varepsilon/2$. By Theorem 1. 1 this is possible. Next we choose N so large that the second term is also less than $\varepsilon/2$. The possibility of this is a consequence of the relation

$$\lim_{N\to\infty} A_N = A$$

which follows from

$$\lim_{N\to\infty} \varphi_N(h) = 0.$$

The preceding argument shows that for every t

$$\lim_{N\to\infty} f_N(t) = 1$$

what was to be proved.

Now, turning to the proof of the general case, let us decompose the function $\varphi_N(h)$ as

$$\varphi_N(h) = \varphi_N^{(1)}(h) + \varphi_N^{(2)}(h)$$

where

$$\varphi_N^{(1)}(h) = \begin{cases} \varphi_N(h) & \text{if} \quad \varphi_N(h) \leq M, \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi_N^{(2)}(h) = \varphi_N(h) - \varphi_N^{(1)}(h)$$

and M is a constant of which we shall dispose later. Denote $f_N^{(1)}(t)$ and $f_N^{(2)}(t)$

the characteristic functions of the random variables

$$\int_A \varphi_N^{(1)}(h)\,\xi(dA) \quad \text{and} \quad \int_A \varphi_N^{(2)}(h)\,\xi(dA),$$

respectively, and consider first the latter.

Let $\{y_k^{(n)}\}$ be an infinitely fine dividing point double sequence for which $\{y_k^{(n)}\} \subset \{y_k^{(n+1)}\}$ and denote by $\{L_{Nk}^{(n)}\}$ the sequence of subdivisions determined by $\varphi_N^{(2)}(h)$ and $\{y_k^{(n)}\}$. If $f_{Nn}^{(2)}(t)$ denotes the characteristic function of the random variable

$$\sum_k y_k^{(n)}\,\xi(AL_{Nk}^{(n)}),$$

then obviously

$$f_{Nn}^{(2)}(t) \Rightarrow f_N^{(2)}(t) \tag{2.40}$$

if $n \to \infty$.

Let furthermore $\{H_k\}$ denote the subdivision corresponding to the dividing point sequence $\{y_k^{(1)}\} = \{y_k\}$ and the function $\varphi(h)$, finally denote by K the following number: $K = \sup_k (y_{k+1} - y_k)$. If $M > 1$, then by applying Theorem 2.6 we get

$$|1-f_{Nn}^{(2)}(t)| \leqq \sum_{k:\,y_k^{(n)} \geqq M} W(y_k^{(n)}t, AL_{Nk}^{(n)}) \leqq \sum_{k:\,y_k^{(1)} \geqq M} W(y_{k+1}^{(1)}t, AL_{Nk}^{(1)}) \leqq$$

$$\leqq \sum_{k:\,y_k^{(1)} \geqq M} W(y_k^{(1)}(K+1)t, AL_{Nk}^{(1)}) \leqq \sum_{k:\,y_k \geqq M} W(y_k(K+1)t, AH_k).$$

Hence

$$|1-f_N^{(2)}(t)| \leqq \sum_{k:\,y_k \geqq M} W(y_k(K+1)t, AH_k). \tag{2.41}$$

Fixing the number t, first we choose M so large that the right-hand member of (2.41) is less than $\varepsilon/2$. The possibility of this is ensured by Theorem 2.6. Next, if N is large enough, then

$$|1-f_N^{(1)}(t)| < \frac{\varepsilon}{2}.$$

If we keep the notation $f_N(t)$ for the characteristic function of the random variable

$$\int_A \varphi_N(h)\,\xi(dA),$$

then since the sets of those h's for which $\varphi_N^{(1)}(h) \neq 0$ and $\varphi_N^{(2)}(h) \neq 0$, respectively, are disjoint, we have

$$f_N(t) = f_N^{(1)}(t)\,f_N^{(2)}(t)$$

whence

$$|1-f_N(t)| \leqq |1-f_N^{(1)}(t)| + |1-f_N^{(2)}(t)| < \varepsilon$$

for large N's. Hence our theorem is proved.

§ 6. The case of a non-negative set function

If the random variables $\xi(A)$ $(A \in \mathfrak{S})$ are non-negative, then the sequence (1.2) converges not only stochastically to the limit (1.3), but also with probability 1. This assertion is a special case of the following theorem which is a strong version of Theorem 2.9:

THEOREM 2.10. *If for every $B \in \mathfrak{S}$ we have $\xi(B) \geqq 0$ and $\varphi(h), \varphi_N(h)$ $(N=1, 2, \ldots)$ are measurable functions such that*

$$\lim_{N\to\infty} \varphi_N(h) = \varphi(h), \quad |\varphi_N(h)| \leqq \psi(h) \qquad (N=1, 2, \ldots)$$

and $\psi(h)$ is u-integrable over the set $A \in \mathfrak{S}$, then the same holds for $\varphi(h), \varphi_N(h)$ $(N=1, 2, \ldots)$ and

$$\lim_{N\to\infty} \int_A \varphi_N(h)\,\xi(dA) = \int_A \varphi(h)\,\xi(dA).$$

PROOF. It is sufficient to consider the case when $\varphi(h) = 0, \varphi_N(h) \geqq 0$ $(N=1, 2, \ldots)$. Let ε be a positive number and A_N the following set:

$$A_N = \{h : h \in A, \varphi_N(h) \geqq \varepsilon\}.$$

Thus we obtain

$$(2.42) \quad \int_A \varphi_N(h)\,\xi(dA) = \int_{A_N} \varphi_N(h)\,\xi(dA) + \int_{A-A_N} \varphi_N(h)\,\xi(dA) \leqq \int_{A_N} \psi(h)\,\xi(dA) + \varepsilon\,\xi(A).$$

Using Theorem 2.4 of the present paper and Theorem 4.3 of [18], we conclude

$$\lim_{N\to\infty} \int_A \varphi_N(h)\,\xi(dA) = 0.$$

Hence our assertion follows.

III. SOME PROPERTIES OF THE INDEFINITE INTEGRAL

§ 1. Derivation of a completely additive set function

Let us suppose that the function $\varphi(h)$ is integrable over all sets of $\mathfrak{S}$ with respect to the completely additive set function $\xi(A)$ and denote by η the indefinite integral:

$$(3.1) \qquad \eta(A) = \int_A \varphi(h)\,\xi(dA) \qquad (A \in \mathfrak{S}).$$

First we prove that the function $\varphi(h)$ is uniquely determined by the set functions ξ and η except at most on a 0-set of the set function ξ. This is expressed by

THEOREM 3. 1. *Let $\varphi_1(h)$ and $\varphi_2(h)$ be two measurable functions integrable over all sets $A \in \mathfrak{S}$ with respect to the set function ξ and suppose that for every $A \in \mathfrak{S}$ we have*

$$\int_A \varphi_1(h)\,\xi(dA) = \int_A \varphi_2(h)\,\xi(dA). \tag{3. 2}$$

Then there exists a set I such that $H - I$ is a 0-set relative to the set function ξ and

$$\varphi_1(h) = \varphi_2(h) \quad \text{if} \quad h \in I.$$

PROOF. Let us consider the function $\psi(h) = \varphi_1(h) - \varphi_2(h)$. According to (3. 2), for every $A \in \mathfrak{S}$ we have

$$\int_A \psi(h)\,\xi(dA) = 0. \tag{3. 3}$$

The fulfilment of this relation obviously does not depend on the concrete representation of the random variables $\xi(A)$ $(A \in \mathfrak{S})$, or more precisely, it must hold also for other stochastic set functions defined on the elements of $\mathfrak{S}$ whenever the so-called finite-dimensional distributions, i. e. the distributions of the vectors $(\xi(A_1), \dots, \xi(A_n))$ $(A_1 \in \mathfrak{S}, \dots, A_n \in \mathfrak{S})$ for the different set functions remain unchanged.

Let us imagine the sample space, where the random variables $\xi(A)$ $(A \in \mathfrak{S})$ are defined, in two exemplars and denote them by Ω_1 and Ω_2, respectively. We shall consider the product space

$$\bar{\Omega} = \Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2)\} \qquad (\omega_1 \in \Omega_1, \omega_2 \in \Omega_2)$$

with the product probability measure. Let

$$\begin{aligned} \xi_1(A) &= \xi_1((\omega_1, \omega_2), A) = \xi(\omega_1, A), \\ \xi_2(A) &= \xi_2((\omega_1, \omega_2), A) = \xi(\omega_2, A). \end{aligned} \tag{3. 4}$$

If $A_1, \dots, A_m$ and $B_1, \dots, B_n$ are arbitrary systems of sets of $\mathfrak{S}$, then the vectors

$$(\xi_1(A_1), \dots, \xi_1(A_m)), \quad (\xi_2(B_1), \dots, \xi_2(B_n))$$

are independent. Hence the set function

$$\xi_3(A) = \xi_1(A) - \xi_2(A) \qquad (A \in \mathfrak{S}) \tag{3. 5}$$

is completely additive and every random variable $\xi_3(A)$ $(A \in \mathfrak{S})$ has a (relative to the point 0) symmetric probability distribution. We know furthermore that

$$\int_A \psi(h)\,\xi_3(dA) = \int_A \psi(h)\,\xi_1(dA) - \int_A \psi(h)\,\xi_2(dA) = 0 - 0 = 0. \tag{3. 6}$$

If A_N is the set of those h's for which

$$\frac{1}{N} \leqq |\psi(h)| \leqq N \qquad (N \geqq 1), \tag{3.7}$$

then obviously $A_N \subseteq A_{N+1}$ $(N=1, 2, \ldots)$.

Let $\{y_k^{(n)}\}$ be an infinitely fine dividing point double sequence and $\{H_k^{(n)}\}$ the sequence of subdivisions determined by $\psi(h)$ and $\{y_k^{(n)}\}$. For a fixed N we have

$$\sum_k y_k^{(n)} \xi_3(A_N H_k^{(n)}) \Longrightarrow 0 \tag{3.8}$$

if $n \to \infty$. Thus the sum in (3.8) satisfies the weak law of large numbers (cf. [7], § 22, Definition 3), hence (cf. [7], § 22, Remark to the Theorem)

$$\lim_{n\to\infty} \sum_k M \left| \frac{(y_k^{(n)} \xi_3(A_N H_k^{(n)}))^2}{1+(y_k^{(n)} \xi_3(A_N H_k^{(n)}))^2} \right| = 0. \tag{3.9}$$

Taking into account the definition of the set A_N, it follows that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \sum_k M \left| \frac{(y_k^{(n)} \xi_3(A_N H_k^{(n)}))^2}{1+(y_k^{(n)} \xi_3(A_N H_k^{(n)}))^2} \right| \\ &\geqq \frac{1}{N^4} \lim_{n\to\infty} \sum_k M \left[\frac{(\xi_3(A_N H_k^{(n)}))^2}{1+(\xi_3(A_N H_k^{(n)}))^2} \right]. \end{aligned} \tag{3.10}$$

Applying again the necessary and sufficient condition of the weak law of large numbers we get

$$\sum_k \xi_3(A_N H_k^{(n)}) \Longrightarrow 0 \tag{3.11}$$

if $n \to \infty$. But for every n (3.11) equals $\xi_3(A_N)$, hence $\xi_3(A_N) = 0$.

Let L denote the set where $\psi(h) \neq 0$. It follows from the precedings that

$$\xi_3(L) = \lim_{N\to\infty} \xi_3(A_N). \tag{3.12}$$

Now suppose that $B \in L\mathfrak{S}$. Since by (3.12)

$$\xi_3(B) + \xi_3(L-B) = 0, \tag{3.13}$$

further $\xi_3(B)$ and $\xi_3(L-B)$ are independent and depend on symmetric distributions (with respect to the point 0), we obtain that $\xi_3(B) = 0$. On the other hand, $\xi_1(B)$ and $\xi_2(B)$ are also independent and $\xi_3(B) = \xi_1(B) - \xi_2(B)$, hence $\xi_1(B) = \text{const.}$

Now we return to our original set function ξ. As $\xi(B)$ and $\xi_1(B)$ depend on the same probability distribution, we conclude that there is a completely

additive number-valued set function μ defined on the σ-algebra $L\mathfrak{S}$ such that

(3.14) $$\xi(B)=\mu(B) \quad \text{for} \quad B\in L\mathfrak{S}.$$

Since

(3.15) $$\int_A \psi(h)\,\mu(dA)=\int_A \psi(h)\,\xi(dA)=0 \quad (A\in L\mathfrak{S}),$$

the Radon—Nikodym theorem implies the existence of a set $I\in L\mathfrak{S}$ such that $L-I$ is a 0-set with regard to μ (or what is the same, to ξ) and

(3.16) $$\psi(h)=0 \quad \text{if} \quad h\in I.$$

This set I fulfils the requirements of our theorem. Q. e. d.

We shall call the function $\varphi(h)$ the derivative of the completely additive set function η relative to the completely additive set function ξ and denote it by

(3.17) $$\varphi(h)=\frac{d\eta}{d\xi}.$$

We may call also the set function η absolutely continuous relative to the set function ξ. In the following § we formulate the chain rule for the derivation of stochastic set functions which is the generalization of the classical Radon—Nikodym theorem.

§ 2. The chain rule for the derivation of completely additive stochastic set functions

In this section we prove the following

THEOREM 3.2. *Let ξ, η, ζ be completely additive stochastic set functions defined on the σ-ring $\mathfrak{S}$. Suppose that there exist measurable functions $\varphi(h)$ and $\psi(h)$ such that for every $A\in\mathfrak{S}$*

(3.18) $$\zeta(A)=\int_A \psi(h)\,\eta(dA), \quad \eta(A)=\int_A \varphi(h)\,\xi(dA).$$

In this case $\varphi(h)\,\psi(h)$ is integrable over all sets of $\mathfrak{S}$ relative to the completely additive set function ξ and

(3.19) $$\zeta(A)=\int_A \varphi(h)\,\psi(h)\,\xi(dA)$$

for every $A\in\mathfrak{S}$. By other words: the derivative of ζ relative to ξ exists and

(3.20) $$\frac{d\zeta}{d\xi}=\frac{d\zeta}{d\eta}\,\frac{d\eta}{d\xi}.$$

PROOF. Let us suppose first that $\varphi(h)\psi(h)$ is a bounded function. We choose an infinitely fine dividing point double sequence $\{y_k^{(n)}\}$ and denote by $\{H_k^{(n)}\}$ the sequence of subdivisions determined by $\psi(h)$ and $\{y_k^{(n)}\}$. Let us define the sequences of functions $\psi_n(h), \varphi_n(h)$ as follows:

$$\psi_n(h) = y_k^{(n)} \quad \text{if} \quad h \in H_k^{(n)} \qquad (k = 0, \pm 1, \pm 2, \ldots;\ n = 1, 2, \ldots),$$
$$\varphi_n(h) = \varphi(h)\psi_n(h) \qquad (n = 1, 2, \ldots).$$

Our assumptions imply

$$\sum_k y_k^{(n)} \eta(AH_k^{(n)}) \Longrightarrow \xi(A). \tag{3.21}$$

On the other hand,

$$\begin{aligned}\sum_k y_k^{(n)} \eta(AH_k^{(n)}) &= \sum_k y_k^{(n)} \int_{AH_k^{(n)}} \varphi(h)\xi(dA) = \sum_k \int_{AH_k^{(n)}} y_k^{(n)} \varphi(h)\xi(dA) = \\ &= \sum_k \int_{AH_k^{(n)}} \varphi(h)\psi_n(h)\xi(dA) = \int_A \varphi_n(h)\xi(dA).\end{aligned} \tag{3.22}$$

Relations (3.21) and (3.22) together imply

$$\xi(A) = \int_A \varphi(h)\psi(h)\xi(dA). \tag{3.23}$$

Now we return to the general case. First we prove that the function $\varphi(h)\psi(h)$ is integrable over all measurable sets relative to the completely additive set function ξ. For this purpose we consider a dividing point sequence $\{y_k\}$ with the corresponding sequence of sets:

$$H_k = \{h:\ y_k \leq \varphi(h)\psi(h) < y_{k+1}\} \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

According to what has been said above concerning the bounded functions, it follows that

$$\xi(CH_k) = \int_{CH_k} \varphi(h)\psi(h)\xi(dA)$$

where C is an arbitrary but fixed element of the σ-ring $\mathfrak{S}$.

Since ξ is a completely additive stochastic set function, the series

$$\sum_{k=-\infty}^{\infty} \xi(CH_k),$$

consequently also the series

$$\sum_{k=-\infty}^{\infty} \int_{CH_k} \varphi(h)\psi(h)\xi(dA)$$

converges with probability 1 regardless of the order of summation.

On the other hand, the relation

$$\varphi(h)\psi(h) = (\varphi(h)\psi(h) - y_k) + y_k$$

implies that

$$(3.24)\qquad \int_{CH_k} \varphi(h)\psi(h)\xi(dA) = \int_{CH_k} (\varphi(h)\psi(h)-y_k)\xi(dA) + y_k\xi(CH_k).$$

If $g_k(t)$ denotes the characteristic function of the first term on the right-hand side, then by Theorem 1.4 we have

$$(3.25)\qquad |1-g_k(t)| \leqq W(\delta t, CH_k)$$

where W is the measure defined by (1.4) with respect to the set function ξ and $\delta = \sup_k(y_{k+1}-y_k)$. It follows from (3.25) that

$$\sum_{k=-\infty}^{\infty} |1-g_k(t)| \leqq \sum_{k=-\infty}^{\infty} W(\delta t, CH_k) = W(\delta t, C) < \infty.$$

Thus the series

$$\sum_{k=-\infty}^{\infty} \int_{CH_k} (\varphi(h)\psi(h)-y_k)\xi(dA)$$

converges with probability 1 regardless of the order of summation. Taking (3.24) into account we conclude that the series

$$\sum_{k=-\infty}^{\infty} y_k\xi(CH_k)$$

does also which implies the integrability of $\varphi(h)\psi(h)$ relative to the set function ξ on the set C. Since C was arbitrary, our assertion holds.

Now let A_N be the following set:

$$A_N = \{h:\ h\in A, -N \leqq \varphi(h)\psi(h) < N\}.$$

Since the integral of $\varphi(h)\psi(h)$ coincides with the corresponding value of ζ on every set where $\varphi(h)\psi(h)$ is bounded, it follows that

$$(3.26)\qquad \zeta(A_N) = \int_{A_N} \varphi(h)\psi(h)\xi(dA).$$

If $N\to\infty$, then the monotonous sequence A_N converges to A. Moreover, since ζ and the indefinite integral of the function $\varphi(h)\psi(h)$ are completely additive set functions, (3.26) implies that

$$(3.27)\quad \zeta(A) = \lim_{N\to\infty} \zeta(A_N) = \lim_{N\to\infty} \int_{A_N} \varphi(h)\psi(h)\xi(dA) = \int_A \varphi(h)\psi(h)\xi(dA).$$

Thus Theorem 3.2 is proved.

REMARK. The classical Radon—Nikodym theorem cannot be generalized word for word to our integral. E. g., let H be the interval [0, 1] and $\mathfrak{S}$ the family of the Lebesgue measurable sets. If $\xi(A)$ and $\eta(A)$ are two completely

additive set functions with the characteristic functions

$$e^{im(A)t - m(A)\frac{t^2}{2}} \quad \text{and} \quad e^{-m(A)\frac{t^2}{2}},$$

respectively, where $m(A)$ is the Lebesgue measure of A, then the relation $\xi(A)=0$ implies that $\eta(A)=0$. Nevertheless, the relation

$$\eta(A)=\int_A \varphi(h)\xi(dA) \qquad (A \in \mathfrak{S})$$

can be fulfilled for no function $\varphi(h)$.

In fact, as it is easy to see, the characteristic function of $\eta(A)$ equals

$$\exp\left\{it\int_A \varphi(h)m(dA)-\frac{t^2}{2}\int_A \varphi^2(h)m(dA)\right\}.$$

For every $A \in \mathfrak{S}$ this must be identical with the function of t

$$\exp\left\{-\frac{t^2}{2}m(A)\right\}$$

which is impossible.

IV. SOME THEOREMS CONCERNING THE EXPECTATIONS AND DISPERSIONS

§ 1. The expectations and dispersions of a completely additive set function

If $\xi(A)$ is an additive set function defined on a ring of sets $\mathfrak{R}$ and for every $A \in \mathfrak{R}$ the expectation

$$M(A)=\mathbf{M}(\xi(A)) \tag{4. 1}$$

exists, then $M(A)$ $(A \in \mathfrak{R})$ is an additive number-valued set function. The same holds for the variances

$$D^2(A)=\mathbf{D}^2(\xi(A)) \tag{4. 2}$$

if they exist. The complete additiveness of the set functions (4. 1) and (4. 2) holds also if $\xi(A)$ is a completely additive set function. This is expressed by

THEOREM 4. 1. *If $\xi(A)$ is a completely additive set function defined on a ring $\mathfrak{R}$ for which the expectations $\mathbf{M}(\xi(A))$ or the dispersions $\mathbf{D}(\xi(A))$ $(A \in \mathfrak{R})$ exist, then, respectively, the set functions* (4. 1) *and* (4. 2) *are completely additive.*

PROOF. Let $A_1, A_2, \dots$ be a sequence of disjoint sets of $\mathfrak{R}$ with $A=\sum_{k=1}^{\infty} A_k \in \mathfrak{R}$. According to a theorem of DOOB (cf. [5], p. 339, Theorem 5. 2),

$$\mathbf{M}(\xi(A))=\sum_{k=1}^{\infty}\mathbf{M}(\xi(A_k))$$

whence

$$M(A)=\sum_{k=1}^{\infty}M(A_k). \tag{4.3}$$

In order to simplify the proof of our assertion relative to the dispersions we suppose that $M(B)=0$ for every $B\in\mathfrak{R}$. By the above-mentioned theorem of DOOB we have also

$$\lim_{n\to\infty}\mathbf{M}\left[\left(\xi(A)-\sum_{k=1}^{n}\xi(A_k)\right)^2\right]=0.$$

Since

$$\xi(A)-\sum_{k=1}^{n}\xi(A_k)=\xi(B_{n+1})$$

where $B_n=\sum_{k=n}^{\infty}A_k$, we have

$$\lim_{n\to\infty}D^2(B_{n+1})=\lim_{n\to\infty}\mathbf{D}^2(\xi(B_{n+1}))=0. \tag{4.4}$$

On the other hand,

$$D^2(A)=\sum_{k=1}^{n}D^2(A_k)+D^2(B_{n+1}) \tag{4.5}$$

whence the theorem follows.

REMARK. *If $\xi(A)$ $(A\in\mathfrak{R})$ is a completely additive set function and for every $A\in\mathfrak{S}$ the dispersion $D^2(A)=\mathbf{D}^2(\xi(A))$ exists, further the completely additive number-valued set functions* (4. 1) *and* (4. 2) *can be extended to $\mathfrak{S}(\mathfrak{R})$, then $\xi(A)$ can also be extended to $\mathfrak{S}(\mathfrak{R})$ and*

$$\mathbf{D}^2(\xi^*(B))=D^{*2}(B),\quad \mathbf{M}(\xi^*(B))=M^*(B)\qquad (B\in\mathfrak{S}(\mathfrak{R})) \tag{4.6}$$

where ξ^*, D^{*2} and M^* denote the extended set functions corresponding to ξ, D^2 and M, respectively.

PROOF. According to Theorem 3. 11 of [18], the completely additive set function $\xi(A)-M(A)$ $(A\in\mathfrak{R})$ can be extended to $\mathfrak{S}(\mathfrak{R})$. If $\mathfrak{M}$ denotes the family of those sets B for which $\xi^*(B)$ has a finite dispersion and (4. 6) holds, then from the well-known theorems concerning the convergence of series of independent random variables it follows that $\mathfrak{M}$ is a monotone class of sets (cf. [8], p. 27). Since $\mathfrak{R}\subset\mathfrak{M}$, we conclude that $\mathfrak{M}=\mathfrak{S}(\mathfrak{R})$ (cf. [8], p. 27, Theorem B). Q. e. d.

§ 2. The expectations and dispersions of the integral (1.3)

In general, the existence of the expectations $\mathbf{M}(\xi(A))$ $(A \in \mathfrak{S})$ (and dispersions $\mathbf{D}^2(\xi(A))$ $(A \in \mathfrak{S})$, resp.) does not imply the existence of the corresponding quantities of the integral (1.3). In order to prove such a theorem we have to make further assumptions. We shall prove two theorems in this direction. In the first one we do not strive to formulate a very general assertion.

THEOREM 4.2. *Let $\xi(A)$ $(A \in \mathfrak{S})$ be a completely additive set function. Suppose that for every $B \in \mathfrak{S}$ the expectation $M(B) = \mathbf{M}(\xi(B))$ exists and there is a random variable ζ such that $\mathbf{M}(|\zeta|) < \infty$ and*

$$\sum_{k=1}^{r} |\xi(A_k)| \leq \zeta$$

where $A_1, \ldots, A_r$ is an arbitrary system of disjoint sets of $\mathfrak{S}$. In this case for every bounded and measurable function $\varphi(h)$ the expectations of the random variables

$$\eta(B) = \int_B \varphi(h)\xi(dA) \tag{4.7}$$

exist and

$$\mathbf{M}(\eta(B)) = \int_B \varphi(h)M(dA) \tag{4.8}$$

where the integral is taken in the sense of Radon.

PROOF. Let $\{y_k^{(n)}\}$ be an infinitely fine dividing point double sequence and K the upper bound of the values of $|\varphi(h)|$. Consider the series

$$\eta_n = \sum_{k=-\infty}^{\infty} y_k^{(n)}\xi(BH_k^{(n)})$$

where $\{H_k^{(n)}\}$ is the sequence of subdivisions determined by $\{y_k^{(n)}\}$ and $\varphi(h)$. Since

$$\eta_n \to \int_B \varphi(h)\xi(dA) \quad \text{if} \quad n \to \infty,$$

moreover

$$|\eta_n| \leq \sum_{k=-\infty}^{\infty} |y_k^{(n)}|\,|\xi(BH_k^{(n)})| \leq K\zeta, \tag{4.9}$$

the bounded convergence theorem of LEBESGUE (cf. [8], § 26, Theorem 4) implies that

$$\lim_{n\to\infty} \mathbf{M}(\eta_n) = \mathbf{M}(\eta(B)). \tag{4.10}$$

In view of (4.9) we have also

$$\mathbf{M}(\eta_n) = \sum_{k=-\infty}^{\infty} y_k^{(n)} \mathbf{M}(\xi(BH_k^{(n)})) = \sum_{k=-\infty}^{\infty} y_k^{(n)} M(BH_k^{(n)}). \tag{4.11}$$

Comparing (4.10) and (4.11), the theorem follows.

THEOREM 4.3. *Let $\xi(A)$ $(A \in \mathfrak{S})$ be a completely additive set function. Suppose that for every $B \in \mathfrak{S}$ the dispersion $D^2(B) = \mathbf{D}^2(\xi(B))$ exists and $\varphi(h)$ is such a measurable function that the integrals*

$$\int_B \varphi(h) M(dA), \quad \int_B \varphi^2(h) D^2(dA)$$

exist. In this case the integral (4.7), $\mathbf{M}(\eta(B))$ *and* $\mathbf{D}^2(\eta(B))$ *also exist,* (4.8) *holds and*

$$\mathbf{D}^2(\eta(B)) = \int_B \varphi^2(h) D^2(dA). \tag{4.12}$$

PROOF. Let $\{y_k^{(n)}\}$ be an infinitely fine dividing point double sequence and $\{H_k^{(n)}\}$ the sequence of subdivisions determined by $\{y_k^{(n)}\}$ and the function $\varphi(h)$. We show that the sequence

$$\zeta_n = \sum_k y_k^{(n)} \xi(BH_k^{(n)})$$

converges in mean. In fact, if $\{z_k\}$ is the union of the dividing point sequences $\{y_k^{(n)}\}, \{y_k^{(m)}\}$ (not indicating the dependence on n and m) and $\{L_k\}$ is the corresponding sequence of subdivisions, then

$$\zeta_n - \zeta_m = \sum_k \sum_{j:L_j \subset H_k^{(n)}} (y_k^{(n)} - z_j)\xi(AL_j) - \sum_k \sum_{j:L_j \subset H_k^{(m)}} (y_k^{(m)} - z_j)\xi(AL_j). \tag{4.13}$$

If $\delta_{kj}^{(n)} = y_k^{(n)} - z_j$, then, since the double sequence $\{y_k^{(n)}\}$ is infinitely fine, it follows

$$\delta^{(n)} = \sup_k \max_{j:L_j \subset H_k^{(n)}} |\delta_{kj}^{(n)}| \to 0 \quad \text{if} \quad n \to \infty. \tag{4.14}$$

Formulae (4.13) and (4.14) imply

$$\begin{aligned} \mathbf{D}^2(\zeta_n - \zeta_m) &\leq 2\sum_k \sum_{j:L_j \subseteq H_k^{(n)}} (y_k^{(n)} - z_j)^2 \mathbf{D}^2(\xi(AL_j)) + \\ &+ 2\sum_k \sum_{j:L_j \subseteq H_k^{(m)}} (y_k^{(m)} - z_j)^2 \mathbf{D}^2(\xi(AL_j)) \leq \\ &\leq 4(\delta^{(n)})^2 \sum_j \mathbf{D}^2(\xi(AL_j)) = 4(\delta^{(n)})^2 \mathbf{D}^2(\xi(A)) \to 0 \quad \text{if} \quad m, n \to \infty \end{aligned} \tag{4.15}$$

and it is easy to see that $\mathbf{M}(\zeta_n - \zeta_m) \to 0$ for $m, n \to \infty$. We know that

$$\zeta_n \Rightarrow \eta(B) \quad \text{if} \quad n \to \infty, \tag{4.16}$$

hence

$$\left.\begin{aligned} &\mathbf{M}(\zeta_n) \to \mathbf{M}(\eta(B)), \\ &\mathbf{M}(\zeta_n^2) \to \mathbf{M}(\eta^2(B)) \end{aligned}\right\} \quad \text{if} \quad n \to \infty. \tag{4. 17}$$

Using the fact that the random variables $\xi(BH_k^{(n)})$ $(k = 0, \pm 1, \pm 2, \ldots)$ are independent, we get

$$\mathbf{D}^2(\zeta_n) = \mathbf{M}(\zeta_n^2) - \mathbf{M}^2(\zeta_n) = \sum_k (y_k^{(n)})^2 D^2(BH_k^{(n)}) \tag{4. 18}$$

whence (4. 11) follows immediately. (4. 8) is equivalent to the first line of (4. 17). Q. e. d.

A similar formula could be deduced for the third central moments. In fact, the third central moment of a sum of independent random variables is equal to the sum of the single third central moments. As this question is not a very significant one, we do not enter into the details.

V. THE CHARACTERISTIC FUNCTIONAL

Let $\mathfrak{B}$ denote the space of those measurable functions $\varphi(h)$ $(h \in H)$ which are almost everywhere bounded relative to the completely additive set function $\xi(A)$ $(A \in \mathfrak{S})$. If we assign to every $\varphi \in \mathfrak{B}$ the norm

$$\|\varphi\| = \operatorname*{vrai\,max}_{h \in H} \varphi(h),$$

then $\mathfrak{B}$ becomes a Banach space.

We consider the functional

$$L(\varphi) = \mathbf{M}\left[\exp i \int_H \varphi(h) \xi(dA)\right] \tag{5. 1}$$

which will be called the *characteristic functional* of the completely additive set function ξ.

The notion of characteristic functional as a generalization of the Fourier integral was introduced by A. N. KOLMOGOROV [15] in the year 1935. For probabilistic applications it was introduced by L. LE CAM [3] in 1947 for stochastic processes and in the same year by S. BOCHNER [1] for random additive set functions. The characteristic functional (5. 1) is neither a special case nor a generalization of the notions introduced by the above-mentioned authors. If the realizations of $\xi(A)$, i. e. the number-valued set functions $\xi(\omega, A)$ for fixed ω's are completely additive, then the functional (5. 1) becomes a special case of the characteristic functional of KOLMOGOROV.

If the functional (5. 1) is known, then we know also the probability situations concerning the random variables $\xi(A)$ $(A \in \mathfrak{S})$. This is expressed more precisely by

THEOREM 5. 1. *The characteristic functional $L(\varphi)$ determines completely the probability measure on the smallest σ-ring relative to which the random variables $\xi(A)\,(A \in \mathfrak{S})$ are measurable.*

PROOF. Let $A_1, \ldots, A_n$ be a system of disjoint sets of $\mathfrak{S}$. We show that $L(\varphi)$ determines the joint characteristic function of the random variables $\xi(A_1), \ldots, \xi(A_n)$. Let us define the functions

$$\varphi_{t_k}(h) = \begin{cases} t_k & \text{if } \; h \in A_k, \\ 0 & \text{otherwise} \end{cases} \qquad (k = 1, \ldots, n).$$

Obviously,

$$(5.2) \qquad \mathbf{M}[\exp i(t_1\xi(A_1) + \cdots + t_n\xi(A_n))] = \mathbf{M}\Big[\exp i \int_H \varphi_{t_1, \ldots, t_n}(h)\xi(dA)\Big]$$

where

$$(5.3) \qquad \varphi_{t_1, \ldots, t_n}(h) = \sum_{k=1}^{n} \varphi_{t_k}(h).$$

If the variables $t_1, \ldots, t_n$ run over the set of the real numbers, then $L(\varphi_{t_1, \ldots, t_n}(h))$ gives the characteristic function of the random vector $(\xi(A_1), \ldots, \xi(A_n))$. These so-called finite-dimensional distributions determine the probability measure on the σ-ring in question. Q. e. d.

If for every $\varphi \in \mathfrak{B}$ the moment

$$(5.4) \qquad L_n(\varphi) = \mathbf{M}\Big[\Big(\int_H \varphi(h)\xi(dA)\Big)^n\Big]$$

exists, then we call the functional (5. 4) the *n-th moment of the completely additive set function* $\xi(A)\,(A \in \mathfrak{S})$. By Taylor expansion we get from (5. 1)

$$(5.5) \qquad L(\varphi) = 1 + iL_1(\varphi) + i^2\frac{L_2(\varphi)}{2!} + \cdots + i^{n-1}\frac{L_{n-1}(\varphi)}{(n-1)!} + i^n\frac{L_n'(\varphi)}{n!}R_n(\varphi)$$

where

$$(5.6) \qquad L_n'(\varphi) = M\Big(\Big|\int_H \varphi(h)\xi(dA)\Big|^n\Big)$$

and $R_n(\varphi)$ is such a functional that $|R_n(\varphi)| \leq 1$.

Now we describe the characteristic functional of some simple type of stochastic completely additive set functions.

1. *Poisson type.* In this case

$$(5.7) \qquad f(t, A) = e^{\lambda(A)(e^{it}-1)} \qquad (A \in \mathfrak{S})$$

where $\lambda(A)$ is a finite measure on the σ-ring $\mathfrak{S}$. It is easy to see that

$$(5.8) \qquad L(\varphi) = \exp\Big\{\int_H (e^{i\varphi(h)} - 1)\,\lambda(dA)\Big\}.$$

We mention that every measurable function $\varphi(h)$ (bounded and not bounded equally) is integrable with respect to a completely additive set function of Poisson type. This is plausible because essentially it is composed of a set of purely discontinuous measures (which are the realizations) with finite numbers of discontinuity points. In this case the realizations are completely additive set functions but it is not difficult to prove the general assertion.

In fact, the convergence with probability 1 of the series

$$\sum_k \xi(AH_k^{(n)})$$

implies that for a fixed n $\xi(AH_k^{(n)}) \neq 0$ for at most a finite number of k's, hence

$$\sum_k y_k^{(n)} \xi(AH_k^{(n)})$$

also converges with probability 1. Here $\{y_k^{(n)}\}$ and $\{H_k^{(n)}\}$ have the usual meaning.

2. *The compound Poisson type.* In this case

$$f(t, A) = \exp\left\{\sum_{k=1}^{\infty} C_k(A)(e^{i\lambda_k t} - 1)\right\} \qquad (A \in \mathfrak{S}), \tag{5.9}$$

where $\lambda_1, \lambda_2, \ldots$ is an additive semi-group of real numbers, further $C_k(A)$ $(k = 1, 2, \ldots)$ and

$$C(A) = \sum_{k=1}^{\infty} C_k(A) \tag{5.10}$$

are finite measures on the σ-ring $\mathfrak{S}$.

The characteristic functional has the form

$$L(\varphi) = \exp\left\{\sum_{k=1}^{\infty} \int_H (e^{i\lambda_k \varphi(h)} - 1)\, C_k(dA)\right\}. \tag{5.11}$$

Such as in 1 we can prove that every measurable function $\varphi(h)$ is integrable with respect to a compound Poisson stochastic set function too.

3. *The Laplace—Gauss type.* In this case

$$f(t, A) = e^{itM(A) - \frac{D^2(A)}{2}t^2} \qquad (A \in \mathfrak{S}) \tag{5.12}$$

where $M(A)\,(A \in \mathfrak{S})$ is a number-valued completely additive set function and $D^2(A)\,(A \in \mathfrak{S})$ is a finite measure.

The characteristic functional has the form

$$L(\varphi) = \exp\left\{it \int_H \varphi(h) M(dA) - \frac{t^2}{2} \int_H \varphi^2(h) D^2 dA)\right\} \tag{5.13}$$

where the integrals on the right-hand side in the exponent are Radon and Lebesgue integrals, respectively. Formula (5.13) can be deduced immediately from Theorem 4.3 too.

4. *The Cauchy type.* A completely additive set function $\xi(A)\ (A \in \mathfrak{S})$ will be called of Cauchy type if

(5.14) $$f(t, A) = e^{iC_1(A)t - C_2(A)|t|} \qquad (A \in \mathfrak{S})$$

where $C_1(A)\ \ (A \in \mathfrak{S})$ is a number-valued completely additive set function and $C_2(A)\ \ (A \in \mathfrak{S})$ is a finite measure.

The characteristic functional has the following form:

(5.15) $$L(\varphi) = \exp\left\{ i \int_H \varphi(h) C_1(dA) - \int_H |\varphi(h)|\ C_2(dA) \right\},$$

where the integrals on the right-hand side in the exponent are Radon and Lebesgue integrals, respectively.

It would be possible to define the functional $L(\varphi)$ in a much more extensive space of the functions φ, e.g. in the space of all functions which are integrable over the set H. This latter space is, however, not a Banach space. On the other hand, Theorem 5.1 shows that our space $\mathfrak{B}$ is extensive enough to characterize the probabilistic situation concerning a completely additive stochastic set function.

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ON STOCHASTIC SET FUNCTIONS. III

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In the present paper we are concerned with two topics: the probability distributions of the random variables consisting of a completely additive stochastic set function and the discontinuities of the realizations by supposing the latters to be completely additive number-valued set functions. Some theorems of Chapter III are analogous to those well-known in the theory of stochastic processes with independent increments (cf. mainly [13]).

There are questions arising in a natural way in the field discussed in this paper which we did not consider in detail such as the representation of a stochastic set function as a sum of an atomless and an atomic part, etc. These problems are not very difficult to solve with well-known methods or with our results.

We refer to the definitions and notations of the previous papers [19], [20] which are used here without any remark. Some new definitions and notations will also be introduced at the place where they are needed.

I. PRELIMINARY REMARKS ON TOTALS

The notion of the "total", which is a generalization of the Burkill integral, was introduced by M. COTLAR and Y. FRENKEL [4]. A special case of this was generalized by the author [22] for functions with values in a Banach algebra. These notions and the theorems proved in connection with them find an extensive application in the present paper.

In this Chapter we formulate some definitions and theorems relative to the totals in a form convenient for our treatment. We do not strive in the definitions for the most general versions and mention only the simplest theorems. To other theorems proved in the paper [22] we refer at the place of their application.

Let H be an abstract set and $\mathfrak{K}$ a semi-ring of sets consisting of some subsets of H. This means the following:

DEFINITION 1.[1] A class of sets $\mathfrak{H}$ is called a semi-ring if for every pair of sets $A \in \mathfrak{H}$, $B \in \mathfrak{H}$ for which $A \subseteq B$ there is a finite system of sets $C_k \in \mathfrak{H}$ $(k = 1, \ldots, r)$ such that

$$B - A = \sum_{k=1}^{r} C_k,$$

further with every $E \in \mathfrak{H}$ and $F \in \mathfrak{H}$ we have also $EF \in \mathfrak{H}$.

The elements of $\mathfrak{H}$ can obviously always be decomposed as sums of disjoint sets belonging to $\mathfrak{H}$. If $A \in \mathfrak{H}$ and

$$A = \sum_{k=1}^{r} A_k$$

where $A_k \in \mathfrak{H}$ $(k = 1, \ldots, r)$, $A_i A_k = 0$ for $i \neq k$, then the system of sets $A_1, \ldots, A_r$ will be called a *subdivision* of the set A. The subdivisions will be denoted by simple or signed letters $\mathfrak{z}$ as $\mathfrak{z} = \{A_1, \ldots, A_r\}$.

If $\mathfrak{z}' = \{A_1', \ldots, A_{r'}'\}$ and $\mathfrak{z}'' = \{A_1'', \ldots, A_{r''}''\}$ are two subdivisions of the set $A \in \mathfrak{H}$ and every set A_k'' is a part of a set $A_i' \in \mathfrak{H}$, then we write $\mathfrak{z}' \sqsubset \mathfrak{z}''$. Every "total" is in a closed connection with a partial ordering relation $\prec$, which orders (partially) the space of the subdivisions. This relation is supposed to be fulfilling the requirement that $\mathfrak{z}' \sqsubset \mathfrak{z}''$ implies $\mathfrak{z}' \prec \mathfrak{z}''$. In § 1 of Chapter II we consider the case when the relation $\prec$ reduces to the relation $\sqsubset$. If H is a metric space, then a possible partial ordering relation, which we denote by the sign $\prec$, is the following: $\mathfrak{z}' \prec \mathfrak{z}''$ if $\max_k d(A_k'') \leq \max_k d(A_k')$.[2]

Let $f(A)$ $(A \in \mathfrak{H})$ be a set function the values of which lie in a commutative Banach algebra $\mathfrak{B}$. The additive total of this function over a set $A \in \mathfrak{H}$ is defined by

DEFINITION 2. If there is a $g \in \mathfrak{B}$ such that to every $\varepsilon > 0$ a subdivision $\mathfrak{z}_\varepsilon$ of the set A can be found with the property

$$\left\| \sum_{k=1}^{r} f(A_k) - g \right\| < \varepsilon \tag{1. 1}$$

for every $\mathfrak{z} = \{A_1, \ldots, A_r\} \succ \mathfrak{z}_\varepsilon$, then the element g will be called the *additive total* of the function f over the set A and will be denoted by

$$\mathsf{S}_A f(dA).$$

For real-valued set functions (i. e. when $\mathfrak{B}$ is the Banach algebra of the real numbers) we define the lower and upper totals as follows:

[1] This notion of semi-rings is due to Á. CSÁSZÁR.

[2] $d(A)$ denotes the diameter of the set A, i. e. $d(A) = \sup_{h_1 \in A,\, h_2 \in A} d(h_1, h_2)$ where $d(h_1, h_2)$ is the distance between the points h_1, h_2.

DEFINITION 3. Consider the sums

$$\sum_{k=1}^{r} f(A_k)$$

where $\mathfrak{z} = \{A_1, \ldots, A_r\}$ is a subdivision of the set $A \in \mathfrak{H}$ and take the suprema and infima

$$\sup_{\mathfrak{z}:\mathfrak{z}>\mathfrak{z}'} \sum_{k=1}^{r} f(A_k), \qquad \inf_{\mathfrak{z}:\mathfrak{z}>\mathfrak{z}'} \sum_{k=1}^{r} f(A_k).$$

The infimum of the suprema and the supremum of the infima — while $\mathfrak{z}'$ runs over the subdivisions of the set A — will be called *the upper and lower totals*, resp., of the function f over the set A. For these values we introduce the notations

$$\overline{\mathsf{S}}_A f(dA) \quad \text{and} \quad \underline{\mathsf{S}}_A f(dA),$$

respectively.

The following theorems can be proved by well-known arguments:

THEOREM 1. 1. *There is at most one $g \in \mathfrak{B}$ fulfilling the requirements in Definition 1.*

THEOREM 1. 2. *The additive total of a function $f(B)$ $(B \in \mathfrak{H})$ exists over the set $A \in \mathfrak{H}$ if and only if for every $\varepsilon > 0$ there is a subdivision $\mathfrak{z}_\varepsilon = \{A_1, \ldots, A_r\}$ of the set A such that*

$$\left\| \sum_{k=1}^{r} f(A_k) - \sum_{k=1}^{r'} f(A'_k) \right\| < \varepsilon, \tag{1.2}$$

provided $\mathfrak{z}' = \{A'_1, \ldots, A'_{r'}\} > \mathfrak{z}_\varepsilon$.

THEOREM 1. 3. *If $A = A_1 + A_2$ where $A_1 \in \mathfrak{H}, A_2 \in \mathfrak{H}, A \in \mathfrak{H}, A_1 A_2 = 0$ and the function $f(B)$ $(B \in \mathfrak{H})$ is totalizable over the set A, then this holds also for the sets A_1, A_2 and*

$$\mathsf{S}_A f(dA) = \mathsf{S}_{A_1} f(dA) + \mathsf{S}_{A_2} f(dA). \tag{1.3}$$

THEOREM 1. 4. *If $f(B)$ $(B \in \mathfrak{H})$ is real-valued and subadditive, i.e.*

$$f(A_1 + A_2) \leqq f(A_1) + f(A_2)$$

whenever $A_1 \in \mathfrak{H}, A_2 \in \mathfrak{H}, A_1 + A_2 \in \mathfrak{H}$, then the total of f exists over every set $A \in \mathfrak{H}$ where f has a finite variation and

$$\mathsf{S}_A f(dA) = \sup_{\{A_1, \ldots, A_r\}} \sum_{k=1}^{r} f(A_k) \tag{1.4}$$

where $\{A_1, \ldots, A_r\}$ runs over the subdivisions of the set A.

In the sequel we suppose that the Banach algebra $\mathfrak{B}$ has a unity element which will be denoted by e. We introduce the notion of the multiplicative total.

Definition 4. If there is a $g \in \mathfrak{B}$ such that to every $\varepsilon > 0$ a subdivision $\mathfrak{z}_\varepsilon$ of the set $A \in \mathfrak{H}$ can be found with the property

$$\left\| \prod_{k=1}^{r} f(A_k) - g \right\| < \varepsilon$$

for every $\mathfrak{z} = \{A_1, \ldots, A_r\} > \mathfrak{z}_\varepsilon$, then the element g will be called the *multiplicative total* of the function f over the set A and will be denoted by

$$\prod_A f(dA).$$

The following theorems are almost trivial:

Theorem 1.5. *There is at most one $g \in \mathfrak{B}$ fulfilling the requirements in Definition 4.*

Theorem 1.6. *The multiplicative total of a function $f(B)$ ($f(B) \in \mathfrak{B}$, $B \in \mathfrak{H}$) exists over the set A if and only if for every ε there is a subdivision $\mathfrak{z}_\varepsilon = \{A_1, \ldots, A_r\}$ of the set A such that*

$$\left\| \prod_{k=1}^{r} f(A_k) - \prod_{k=1}^{r'} f(A'_k) \right\| < \varepsilon,$$

provided $\mathfrak{z}' = \{A'_1, \ldots, A'_{r'}\} > \mathfrak{z}_\varepsilon$.

In the present paper multiplicative totals are used only in the case when H is a compact metric space, further $\mathfrak{H}$ has the additional properties:

α) If $h \in H$, then $\{h\} \in \mathfrak{H}$.

β) If $A \in \mathfrak{H}$, then for every $\varepsilon > 0$ there exists a subdivision $\mathfrak{z} = \{A_1, \ldots, A_r\}$ of the set A such that

$$\max_k d(A_k) \leq \varepsilon.$$

γ) Relation $<$ reduces to $\prec$.

If the (partial) ordering relation $<$ reduces to the relation $\prec$ and $\mathfrak{H}$ is the semi-ring of the subintervals of an interval $[a, b]$ (permitting open, closed and semi-closed intervals equally), finally if $\mathfrak{B}$ is the Banach algebra of the real numbers, then the additive total introduced by Definition 2 coincides with the Burkill integral.

II. THE PROBABILITY DISTRIBUTION OF A COMPLETELY ADDITIVE STOCHASTIC SET FUNCTION

§ 1. Atomless set functions

One proves in the theory of stochastic processes with independent increments that if some continuity conditions are fulfilled, e. g. if for every $\varepsilon > 0$

$$\mathbf{P}(|\xi_{s+\Delta s} - \xi_s| > \varepsilon) \to 0 \quad \text{if} \quad \Delta s \to 0$$

uniformly in s, then the differences $\xi_{s_2} - \xi_{s_1}$ are distributed according to an infinitely divisible probability distribution. This is the starting point of the considerations on other questions concerning the distributions of the process ξ_s. As the space H is abstract, an analogous continuity condition for stochastic set functions cannot be formulated. There is, however, a property which is at the same time of probabilistic and set-theoretic nature and which enables the proof of some theorems concerning the probability distributions of a completely additive stochastic set function $\xi(A)$: the atomlessness. Its definition is the following:

Definition 5. Let $\xi(B)$ be a completely additive set function defined on a σ-ring $\mathfrak{S}$. A set $A \in \mathfrak{S}$ will be called an *atom* relative to the set function ξ if for every $C \in A\mathfrak{S}$ we have either $\xi(C) = 0$ or $\xi(C) = \xi(A)$.

Definition 6. The completely additive set function $\xi(B)$ $(B \in \mathfrak{S})$ will be called *atomless* if for every atom A we have $\xi(A) = 0$.

The set function $\xi(B)$ $(B \in \mathfrak{S})$ is atomless if and only if for every $A \in \mathfrak{S}$ satisfying

$$\mathbf{P}(\xi(A) \neq 0) > 0$$

there exist sets $A_1 \in A\mathfrak{S}$, $A_2 \in A\mathfrak{S}$, $A_1 A_2 = 0$ such that

$$\mathbf{P}(\xi(A_1) \neq 0) > 0, \quad \mathbf{P}(\xi(A_2) \neq 0) > 0. \tag{2.1}$$

This shows that the values of an atomless set function in a set $A \in \mathfrak{S}$ are not concentrated with probability 1, i. e. they are well-distributed in "every part" of the set A. If for every $A \in \mathfrak{S}$ the random variable $\xi(A)$ is constant, then our definitions 5—6 reduce to the analogous definitions formulated for number-valued set functions.[3]

The following two theorems have a fundamental role in our discussion:

Theorem 2. 1. *Let $\xi(A)$ be a completely additive set function defined on a σ-ring $\mathfrak{S}$. If there is a $T > 0$ such that the measure $W(T, A)$ $(A \in \mathfrak{S})$[4] is atomless, then the same holds for $\xi(A)$ $(A \in \mathfrak{S})$.*

[3] Cf. [10], p. 48.

[4] See e. g. [20], p. 12.

Conversely, if $\xi(A)$ $(A \in \mathfrak{S})$ *is atomless, then for every* $T > 0$ *the measure* $W(T, A)$ $(A \in \mathfrak{S})$ *has also this property.*

Proof. If for some $T > 0$ and $X \in \mathfrak{S}$ we have $W(T, X) = 0$, then $\xi(X) = 0$ which proves the first assertion.

Let us consider the second statement. If $\xi(B)$ $(B \in \mathfrak{S})$ is atomless and $A \in \mathfrak{S}$ is an arbitrary set, then there can be found sets $A_1 \in \mathfrak{S}, A_2 \in \mathfrak{S}$ such that $A_1 A_2 = 0$ and (2. 1) holds. In this case for every T we have

$$W(T, A_1) > 0, \quad W(T, A_2) > 0, \qquad \text{q. e. d.}$$

Theorem 2. 2. *If* $\xi(B)$ $(B \in \mathfrak{S})$ *is an atomless completely additive set function, then for every set* $A \in \mathfrak{S}$ *the distribution of the random variable* $\xi(A)$ *is infinitely divisible.*

Proof. Let T be a fixed positive number. Since the measure $W(T, B)$ $(B \in \mathfrak{S})$ is atomless, to every $\varepsilon > 0$ there is a subdivision of the set A into pairwise disjoint sets $B_1, \ldots, B_r$ belonging to $\mathfrak{S}$ such that

$$W(T, B_k) \leq \varepsilon \qquad (k = 1, \ldots, r).$$

Hence

$$\sup_{|t| \leq T} |1 - f(t, B_k)| \leq \varepsilon \qquad (k = 1, \ldots, r). \tag{2. 2}$$

It follows from the inequality

$$\frac{1}{2T} \int_{-T}^{T} |1 - f(t, B_k)| \, dt \geq \frac{1}{10} \mathbf{P}\left(|\xi(B_k)| > \frac{1}{T} \right)$$

(cf. [19], p. 220) that there exists a sequence of subdivisions $\{B_1^{(n)}, \ldots, B_{k_n}^{(n)}\}$ of the set A such that the random variables in the double sequence

$$\xi(B_1^{(n)}), \ldots, \xi(B_{k_n}^{(n)}) \qquad (n = 1, 2, \ldots)$$

are infinitesimal (cf. [9], § 20). But for every n

$$\xi(A) = \sum_{k=1}^{k_n} \xi(B_k^{(n)}),$$

hence the distribution of the random variable $\xi(A)$ is a limiting distribution of sums of infinitesimal independent random variables. According to Theorem 2 of [9], § 24, this proves our statement.

Let us consider the Lévy's canonical form of the characteristic function of $\xi(A)$:

$$\begin{gathered}\log f(t, A) = i\gamma(A)t - \frac{\sigma^2(A)t^2}{2} + \\ + \int_{(-\infty, 0)} \left(e^{itx} - 1 - \frac{itx}{1 + x^2} \right) dM(x, A) + \int_{(0, \infty)} \left(e^{itx} - 1 - \frac{itx}{1 + x^2} \right) dN(x, A).\end{gathered} \tag{2. 3}$$

A slightly modified form of this is the following:

$$
\begin{aligned}
(2.4)\qquad \log f(t, A) = i\gamma(\tau, A)t - \frac{\sigma^2(A)t^2}{2} + \\
+ \int_{(-\infty,-\tau)} (e^{itx}-1)\,dM(x, A) + \int_{(\tau,\infty)} (e^{itx}-1)\,dN(x, A) + \\
+ \int_{[-\tau,0)} (e^{itx}-1-itx)\,dM(x, A) + \int_{(0,\tau]} (e^{itx}-1-itx)\,dN(x, A)
\end{aligned}
$$

where τ is an arbitrary positive number.

The numbers $\gamma(A), \sigma^2(A), \gamma(\tau, A)$ are uniquely determined by the probability distribution of $\xi(A)$. The same holds for the functions $M(x, A)$, $N(x, A)$ if in the points of discontinuity we make some convention. We shall suppose $M(x, A)$ and $N(x, A)$ to be continuous to the left. From what has been said above and from the convergence theorems of infinitely divisible distributions it follows

THEOREM 2. 3. *Let $\xi(A)$ $(A \in \mathfrak{S})$ be an atomless completely additive set function and τ a fixed positive number. Then the number-valued set functions $\gamma(A), \gamma(\tau, A)$ are completely additive, and $\sigma^2(A), M(x, A)$ (for fixed $x<0$), $-N(x, A)$ (for fixed $x>0$) are finite measures on the σ-ring $\mathfrak{S}$. Each of these number-valued set functions is atomless.*

PROOF. The preceding set functions are obviously additive. Let $B_1, B_2, \ldots$ be a non-increasing sequence of sets of $\mathfrak{S}$ for which $\prod_{k=1}^{\infty} B_k = 0$. Since

$$f(t, B_k) \Longrightarrow 1 \quad \text{if} \quad k\to\infty,$$

it follows that

$$\gamma(B_k)\to 0, \quad \sigma^2(B_k)\to 0,$$
$$M(x, B_k)\to 0 \quad (x<0), \quad N(x, B_k)\to 0 \quad (x>0)$$

if $k\to\infty$.

As for the sequence $\gamma(\tau, B_k)$, we proceed in the following way. The functions $M(x, B_k), N(x, B_k)$ $(k=1, 2, \ldots)$ are monotone, hence the set of their points of discontinuity is countable. Thus there exists a $\tau_1>0$ so that $\tau_1<\tau$, moreover the points $-\tau_1$ and τ_1 are points of continuity of the functions $M(x, B_k)$ $(k=1, 2, \ldots)$ and $N(x, B_k)$ $(k=1, 2, \ldots)$, respectively. But we know that $\lim_{k\to\infty} \gamma(\tau_1, B_k)=0$ (cf. [9], § 19, Theorem 2), hence the relation

$$\gamma(\tau, B_k) = \gamma(\tau_1, B_k) - \int_{\tau_1<x\le\tau} x\,dN(x, B_k) - \int_{-\tau\le x<-\tau_1} x\,dM(x, B_k)$$

proves the statement.

We prove finally that the mentioned set functions are atomless. Let $A \in \mathfrak{S}$ and choose a sequence of subdivisions $\mathfrak{Z}_n = \{B_1^{(n)}, \ldots, B_{k_n}^{(n)}\}$ for which

$$W(T, B_k^{(n)}) \leq \frac{1}{n} \qquad (k = 1, \ldots, k_n;\ n = 1, 2, \ldots). \tag{2.5}$$

If one of the above set functions had an atom, then in view of (2.5) it would be a contradiction. In fact, (2.5) implies that for any sequence $k^{(n)} (1 \leq k^{(n)} \leq k_n)$

$$\xi(B_{k^{(n)}}^{(n)}) \Longrightarrow 0 \quad \text{if} \quad n \to \infty$$

and hence

$$\gamma(B_{k^{(n)}}^{(n)}) \to 0, \quad \sigma^2(B_{k^{(n)}}^{(n)}) \to 0, \quad \gamma(\tau, B_{k^{(n)}}^{(n)}) \to 0 \quad (\text{for } \tau > 0),$$
$$M(x, B_{k^{(n)}}^{(n)}) \to 0 \quad (\text{for } x < 0), \quad N(x, B_{k^{(n)}}^{(n)}) \to 0 \quad (\text{for } x > 0)$$

if $n \to \infty$. Thus Theorem 2.3 is proved.

In the following theorem we establish some connections between the above set functions and the distributions $F(x, A)$ $(A \in \mathfrak{S})$. The totals in this § are taken relative to the (partial) ordering relation $\sqsubset$.

THEOREM 2.4. *If $\xi(B)$ $(B \in \mathfrak{S})$ is an atomless completely additive set function, then for every $A \in \mathfrak{S}$ we have the following relations:*

$$\log f(t, A) = \mathsf{S}_A (f(t, dB) - 1), \tag{2.6}$$

$$\begin{cases} M(x, A) = \mathsf{S}_A F(x, dB) & (x < 0), \\ N(x, A) = \mathsf{S}_A (F(x, dB) - 1) & (x > 0) \end{cases} \tag{2.7}$$

for every x where the functions on the left-hand side are continuous. If $\tau > 0$ is a number such that $M(x, A)$ is continuous at $-\tau$ and $N(x, A)$ at τ, then

$$\gamma(\tau, A) = \mathsf{S}_A \alpha(\tau, dB) \quad \textit{where} \quad \alpha(\tau, B) = \int_{|x| < \tau} x\, dF(x, B), \tag{2.8}$$

finally

$$\sigma^2(A) = \lim_{\varepsilon \to 0} \overline{\mathsf{S}}_A \beta(\varepsilon, dB) = \lim_{\varepsilon \to 0} \underline{\mathsf{S}}_A \beta(\varepsilon, dB) \tag{2.9}$$

where

$$\beta(\varepsilon, B) = \int_{|x| < \varepsilon} x^2 dF(x, B) - \left(\int_{|x| < \varepsilon} x\, dF(x, B) \right)^2.$$

PROOF. First we prove the relation (2.6). Let $\mathfrak{Z} = \{B_1, \ldots, B_r\}$ be a subdivision of the set $A \in \mathfrak{S}$ for which (2.2) holds $\left(0 < \varepsilon < \frac{1}{2}\right)$. Hence we get

$$\left| \log f(t, A) - \sum_{k=1}^{r} (f(t, B_k) - 1) \right| \leq$$
$$\leq \sum_{k=1}^{r} | \log f(t, B_k) - (f(t, B_k) - 1) | \leq \sum_{k=1}^{r} | f(t, B_k) - 1 |^2 \leq W(T, A)\varepsilon. \tag{2.10}$$

If we substitute $\mathfrak{z}' \sqsupset \mathfrak{z}$ for $\mathfrak{z}$, then (2. 10) remains true, hence (2. 6) is proved. It can be seen from (2. 10) that the convergence to the total is uniform in every finite t-interval.

Now, we consider the relations (2. 7)—(2. 8). Let $\mathfrak{z}_1 \sqsubset \mathfrak{z}_2 \sqsubset \dots$ be a sequence of subdivisions of the set A such that

$$\log f(t, A) = \lim_{n\to\infty} \sum_{k=1}^{k_n} (f(t, B_k^{(n)}) - 1) \tag{2. 11}$$

where $\mathfrak{z}_n = \{B_1^{(n)}, \dots, B_{k_n}^{(n)}\}$. Suppose that either (2. 7) or (2. 8) does not hold. Let this be e. g. the first row of (2. 7), the others can be treated similarly. By condition there exists an $x > 0$, an $\varepsilon_0 > 0$ and for every n a subdivision $\mathfrak{z}_n' = \{C_1^{(n)}, \dots, C_{s_n}^{(n)}\} \sqsupset \mathfrak{z}_n$ such that $M(x, A)$ is continuous at the point x and

$$\left| M(x, A) - \sum_{k=1}^{s_n} F(x, C_k^{(n)}) \right| \geq \varepsilon_0 . \tag{2. 12}$$

On account of (2. 11) we get

$$\log f(t, A) = \lim_{n\to\infty} \sum_{k=1}^{s_n} (f(t, C_k^{(n)}) - 1),$$

or otherwise expressed

$$\exp \sum_{k=1}^{s_n} (f(t, C_k^{(n)}) - 1) \Longrightarrow f(t, A) \quad \text{if} \quad n \to \infty . \tag{2. 13}$$

On the left-hand side of (2. 13) there stands a sequence of infinitely divisible characteristic functions. To the members of this sequence in LÉVY's formula correspond the following functions:

$$M_n(x) = \sum_{k=1}^{s_n} F(x, C_k^{(n)}). \tag{2. 14}$$

Relation (2. 13) implies that at every point of continuity of $M(x, A)$

$$\lim_{n\to\infty} M_n(x) = M(x, A)$$

which contradicts (2. 12).

Concerning the proof of relation (2. 8) we remark that in this case we have to use LÉVY's formula in the form (2. 4).

Finally we prove (2. 9). Let us construct a sequence of subdivisions $\mathfrak{z}_n = \{B_1^{(n)}, \dots, B_{k_n}^{(n)}\}$ for the members of which (2. 2) holds and

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \beta(\varepsilon, B_k^{(n)}) = \overline{\mathfrak{S}}_A \beta(\varepsilon, dB). \tag{2. 15}$$

By Theorem 1 of [9], § 22 we have

$$\lim_{\varepsilon\to 0} \overline{\mathfrak{S}}_A \beta(\varepsilon, dB) = \sigma^2(A).$$

A similar argument shows the statement relative to the lower total. Thus our theorem is proved.

§ 2. The probability distribution of a completely additive set function in case of a metric space

In this § we suppose that H is a compact metric space and $\mathfrak{K}$ is a semi-ring of some subsets of H satisfying Conditions α), β) and γ) on p. 378. We do not suppose that the stochastic set function $\xi(A)$ is defined on a σ-ring but only that it is defined on every element of $\mathfrak{K}$. We assume that $\xi(A)$ is completely additive on $\mathfrak{K}$, i. e. besides the independence property[5]

$$\xi(A) = \sum_{k=1}^{\infty} \xi(A_k)$$

whenever $A_k \in \mathfrak{K}$ $(k=1, 2, \ldots)$, $A_i A_k = 0$ for $i \neq k$ and $A = \sum_{k=1}^{\infty} A_k \in \mathfrak{K}$.

Before turning to the considerations on distributions we prove an auxiliary theorem.

THEOREM 2. 5. *If $\xi(A)$ $(A \in \mathfrak{K})$ is a completely additive set function defined on the semi-ring $\mathfrak{K}$, then there exists a completely additive set function $\bar{\xi}(A)$ defined on $\mathfrak{R} = \mathfrak{R}(\mathfrak{K})$[6] for which*

$$\bar{\xi}(A) = \xi(A) \quad \textit{if} \quad A \in \mathfrak{K}.$$

If $\xi(A)$ $(A \in \mathfrak{K})$ satisfies an extension condition formulated in [19], *Chapter III (substituting $\mathfrak{K}$ for $\mathfrak{R}$), then $\bar{\xi}(A)$ $(A \in \mathfrak{R}(\mathfrak{K}))$ does also, i. e. the set function $\bar{\xi}$ can be extended to $\mathfrak{S}(\mathfrak{R})$.*

PROOF. The ring $\mathfrak{R}(\mathfrak{K})$ consists of finite sums of sets belonging to $\mathfrak{K}$. If $A_1, \ldots, A_r$ are disjoint sets of $\mathfrak{K}$ and $A = \sum_{k=1}^{r} A_k$, then put

$$\bar{\xi}(A) = \sum_{k=1}^{r} \xi(A_k).$$

It is easy to see that the definition of the set function $\bar{\xi}$ is unique and it is completely additive on $\mathfrak{R}$.

[5] To disjoint sets there belong independent random variables.
[6] $\mathfrak{R}(\mathfrak{K})$ denotes the smallest ring containing the semi-ring $\mathfrak{K}$.

We prove the second assertion by the aid of Theorem 3. 3 of [19]. If the condition of this theorem holds for $\xi(A)$ $(A \in \mathfrak{K})$, then it obviously holds also for $\xi(A)$ $(A \in \mathfrak{R})$. On the other hand, if some extension condition holds for $\xi(A)$ $(A \in \mathfrak{K})$, then — as it is very easy to see in every special case — the condition of Theorem 3. 3 is satisfied too. Thus ξ can be extended to $\mathfrak{S}(\mathfrak{R})$ which implies the fulfilment of all extension conditions for ξ. Q. e. d.

The fact that for $\xi(A)$ $(A \in \mathfrak{K})$ one of the extension conditions of [19], Chapter III holds, will be mentioned simply as follows: $\xi(A)$ $(A \in \mathfrak{K})$ can be extended to $\mathfrak{S}(\mathfrak{R}) = \mathfrak{S}(\mathfrak{R}(\mathfrak{K}))$. In the following theorems the totals are taken with respect to the (partial) ordering relation $\prec$.

THEOREM 2. 6. *Let $\xi(A)$ $(A \in \mathfrak{K})$ be a completely additive set function and suppose that it can be extended to $\mathfrak{S}(\mathfrak{R})$. In this case for every $A \in \mathfrak{K}$ the following totals exist:*

$$\log g(t, A) = \mathfrak{S}_A (f(t, dB) - 1) \tag{2. 16}$$

where t is an arbitrary but fixed real number;

$$M(x, A) = \mathfrak{S}_A F(x, dB) \qquad (x < 0), \tag{2. 17}$$

$$N(x, A) = \mathfrak{S}_A (F(x, dB) - 1) \qquad (x > 0) \tag{2. 18}$$

except at most a countable x-set and

$$\gamma(\tau, A) = \mathfrak{S}_A \alpha(\tau, dB) \quad \textit{where} \quad \alpha(\tau, B) = \int_{|x| < \tau} x\, dF(x, B). \tag{2. 19}$$

In (2. 19) *τ is a positive number with the property that $M(x, A)$ and $N(x, A)$ are continuous at $-\tau$ and τ, respectively.*

The sequence approximating the total (2. 16) *converges to its limit uniformly in every finite t-interval.*

PROOF. Let $\mathfrak{B}_T$ denote the Banach algebra of the continuous complex-valued functions defined in the interval $[-T, T]$. Considering the functions $f(t, B)$ $(B \in \mathfrak{K})$ only for $|t| \leq T$, we get elements of $\mathfrak{B}_T$. By condition $\xi(A)$ $(A \in \mathfrak{K})$ can be extended to $\mathfrak{S}(\mathfrak{R})$. To the extended set function there corresponds a measure $W(T, A)$ $(A \in \mathfrak{S}(\mathfrak{R}))$. If

$$\delta(T, C) = \sup_{|t| \leq T} |1 - f(t, C)|,$$

then obviously

$$\mathrm{Var}_\delta(B) \leq W(T, B), \tag{2. 20}$$

hence the set function $f(t, C) - 1$ $(|t| \leq T, C \in \mathfrak{K})$, the values of which are taken from $\mathfrak{B}_T$, is of bounded variation and r-continuous.[7] Hence by Theo-

[7] Cf. [22], p. 109, Definition 5.

rem 1 of [22] the total (2.16) exists for every t satisfying $-T \leq t \leq T$ and the convergence is uniform in the interval $[-T, T]$. Since T was an arbitrary positive number, the statement concerning the totals (2.16) follows.

Let $\mathfrak{Z}_n = \{B_1^{(n)}, \ldots, B_{k_n}^{(n)}\}$ be a sequence of subdivisions of the set A such that $\mathfrak{Z}_n \prec \mathfrak{Z}_{n+1}$ and $\max_k d(B_k^{(n)}) \to 0$ if $n \to \infty$. If

(2.21) $$g_n(t, A) = \prod_{k=1}^{k_n} \exp\left(f(t, B_k^{(n)}) - 1\right),$$

then, by the precedings,

(2.22) $$g_n(t, A) \Longrightarrow g(t, A) = \exp \mathfrak{S}_A (f(t, dB) - 1).$$

The characteristic functions $g_n(t, A)$ are infinitely divisible. In Lévy's canonical form (of the type (2.4)) in the place of $M(x)$, $N(x)$ and $\gamma(\tau)$ there stand the following functions:

(2.23) $$\sum_{k=1}^{k_n} F(x, B_k^{(n)}), \quad \sum_{k=1}^{k_n} (F(x, B_k^{(n)}) - 1)$$

and the constant

(2.24) $$\sum_{k=1}^{k_n} \int_{-\tau}^{\tau} x\,dF(x, B_k^{(n)}),$$

respectively. The well-known convergence theorems relative to infinitely divisible distributions (cf. [9], § 19) imply our statements. Q. e. d.

Remark 1. In the same way as we have proved the relevant assertion of Theorem 2.3, we can prove also here that

(2.25) $$\sigma^2(A) = \lim_{\varepsilon \to 0} \overline{\mathfrak{S}}_A \beta(\varepsilon, dB) = \lim_{\varepsilon \to 0} \underline{\mathfrak{S}}_A \beta(\varepsilon, dB)$$

where

$$\beta(\varepsilon, B) = \int_{|x|<\varepsilon} x^2 dF(x, B) - \left(\int_{|x|<\varepsilon} x\,dF(x, B)\right)^2.$$

Remark 2. If we consider the functions $g(t, B)$, $f(t, B)$ $(B \in \mathfrak{K})$ only in the interval $[-T, T]$, then by Theorem 1 of [22]

(2.26) $$\mathrm{Var}_g(B) \leq \mathrm{Var}_{f-1}(B) \qquad (B \in \mathfrak{K})$$

whence

(2.27) $$\sup_{|t| \leq T} |\log g(t, B)| \leq W(T, B) \qquad (B \in \mathfrak{K})$$

where $W(T, B)$ is the same measure as in the proof of Theorem 2.6. Hence by simple arguments follows that for every $\tau > 0$, $\gamma(\tau, A)$ is a bounded, completely additive number-valued set function and $\sigma^2(A)$, $M(x, A)$ $(x < 0)$, $-N(x, A)$ $(x > 0)$ are bounded measures on $\mathfrak{K}$.

THEOREM 2.7. *Let us consider the infinitely divisible characteristic function*

$$g(t, A) = \exp\{i\gamma(\tau, A)t - \frac{\sigma^2(A)t^2}{2}$$

$$+ \int_{(-\infty, -\tau)} (e^{itx}-1)\,dM(x, A) + \int_{(\tau, \infty)} (e^{itx}-1)\,dN(x, A) + \tag{2.28}$$

$$+ \int_{[-\tau, 0)} (e^{itx}-1-itx)\,dM(x, A) + \int_{(0, \tau]} (e^{itx}-1-itx)\,dN(x, A)\},$$

where $M(x, A)$, $N(x, A)$, $\gamma(\tau, A)$, $\sigma^2(A)$ *are defined by formulae* (2.17)—(2.19) *and* (2.25). *Then for every* $B \in \mathfrak{H}$ *and every* t *we have*

$$f(t, B) = \prod\nolimits_B (1 + \log g(t, dA)) \tag{2.29}$$

and the convergence to the total is uniform in every finite t-interval.

PROOF. This theorem follows immediately from formula (2.26) and from Theorem 3 of [22].

Formula (2.29) gives the general form of the probability distributions of the random variables $\xi(A)$ $(A \in \mathfrak{H})$.

According to Theorems 3—4 of [22], the correspondence between the set of distributions $\{F(x, A), A \in \mathfrak{H}\}$ and the set of real-valued set functions $\{M(x, A)\ (x < 0), N(x, A)\ (x > 0), \gamma(\tau, A)\ (\tau > 0), \sigma^2(A), A \in \mathfrak{H}\}$ is one-to-one.

§ 3. The probability distributions in case of a metric space and a weak continuity

In the beginning of this Chapter we have mentioned that in general a weak continuity cannot be formulated for abstract stochastic set functions. It is, however, possible if H is a metric space.

We suppose in this § (such as in § 2) that H is a compact metric space and $\mathfrak{H}$ a semi-ring of some subsets of H satisfying Conditions α), β) and γ) on p. 378. We make here a little digression by omitting the condition of complete additivity and supposing only additivity for the set function $\xi(A)$ $(A \in \mathfrak{H})$. A remarkable special case of this type of stochastic set functions is that generated by the differences of a stochastic process with independent increments ξ_s. In this case H is an interval $[a, b]$, $\mathfrak{H}$ is the semi-ring of all subintervals of $[a, b]$ (permitting closed, open, semi-closed intervals equally) and $\xi(A)$ equals $\xi_{s_2} - \xi_{s_1}$ if $a \leq s_1 \leq s_2 \leq b$ and A equals one of the intervals (s_1, s_2), $[s_1, s_2)$, $(s_1, s_2]$, $[s_1, s_2]$. Our notion of weak continuity (given by Definition 6) reduces in this case to the classical one.

Definition 7. An additive set function $\xi(A)$ defined on the semi-ring $\mathfrak{K}$ is said to be weakly continuous if for every sequence of sets $A_1, A_2, \ldots$ satisfying $\lim_{n\to\infty} d(A_n) = 0$ we have

$$\xi(A_n) \Longrightarrow 0 \quad \text{if} \quad n \to \infty. \tag{2.30}$$

In the following theorems the totals are taken with respect to the (partial) ordering relation $\prec$.

Theorem 2.8. *Let $\xi(A)$ be an additive and weakly continuous set function defined on the semi-ring $\mathfrak{K}$. In this case every random variable $\xi(A)$ $(A \in \mathfrak{K})$ depends on an infinitely divisible distribution and in Lévy's canonical form the functions and constants are the followings:*

$$M(x, A) = \mathfrak{S}_A F(x, dB) \qquad (x < 0), \tag{2.31}$$

$$N(x, A) = \mathfrak{S}_A (F(x, dB) - 1) \qquad (x > 0), \tag{2.32}$$

$$\gamma(\tau, A) = \mathfrak{S}_A \alpha(\tau, dB) \quad \textit{where} \quad \alpha(\tau, B) = \int_{|x|<\tau} x\, dF(x, B), \tag{2.33}$$

finally

$$\sigma^2(A) = \lim_{\varepsilon\to 0} \overline{\mathfrak{S}}_A \beta(\varepsilon, dB) = \lim_{\varepsilon \to 0} \underline{\mathfrak{S}}_A \beta(\varepsilon, dB) \tag{2.34}$$

where

$$\beta(\varepsilon, B) = \int_{|x|<\varepsilon} x^2 dF(x, B) - \left(\int_{|x|<\varepsilon} x\, dF(x, B) \right)^2.$$

Relations (2.31) *and* (2.32) *hold at the points of continuity of the functions $M(x, A)$ and $N(x, A)$, respectively. τ denotes a positive number which has the property that $M(x, A)$ and $N(x, A)$ are continuous at $-\tau$ and τ, respectively.*

Proof. Let $\mathfrak{z}_n = \{B_1^{(n)}, \ldots, B_{k_n}^{(n)}\}$ be a sequence of subdivisions of the set A such that $\mathfrak{z}_n \prec \mathfrak{z}_{n+1}$ and $\lim_{n\to\infty} \max_k d(B_k^{(n)}) = 0$. The weak continuity of $\xi(B)$ $(B \in \mathfrak{K})$ implies that in the double sequence

$$\xi(B_1^{(n)}), \ldots, \xi(B_{k_n}^{(n)}) \tag{2.35}$$

there stand infinitesimal random variables (cf. [9], § 20). Since for every n

$$\xi(A) = \sum_{k=1}^{k_n} \xi(B_k^{(n)}),$$

it follows that (cf. [9], § 24, Theorem 2) the probability distribution of $\xi(A)$ is infinitely divisible. The other part of the theorem follows from Theorem 4 of [9], § 25. Q. e. d.

Contrary to Theorem 2.6 of the preceding §, in this case the total

$$\mathfrak{S}_A (f(t, dB) - 1) \tag{*}$$

does not exist in general. But it can be proved that

$$\log f(t, A) = \mathsf{S}_A (f_1(t, dB) - 1)$$

where

$$f_1(t, B) = f(t, B) e^{-i\alpha(\tau, B) t}, \quad \alpha(\tau, B) = \int_{|x| < \tau} x\, dF(x, B)$$

(cf. the proof of Theorem 1 of [9], § 24).

If $\xi(A)$ is a set function generated by the differences of a stochastic process with independent increments ξ_s, then the totals in Theorem 2.8 reduce to Burkill integrals. For such set functions $\xi(A)$ simple examples can be given where the total (∗) does not exist. We have only to put $\xi_s \equiv \varphi(s)$ where $\varphi(s)$ is a conveniently chosen continuous real function of unbounded variation.

The k-th moment and the k-th semi-invariant of a random variable $\xi(A)$ will in the sequel be denoted by $M_k(A)$ and $\varkappa_k(A)$, respectively:

$$M_k(A) = \int_{-\infty}^{\infty} x^k\, dF(x, A), \quad \varkappa_k(A) = \frac{1}{i^k} \frac{d^k}{dt^k} \log f(t, A) \Big|_{t=0},$$

provided that the integral on the right-hand side of the first relation exists. We prove a theorem concerning their connection.

THEOREM 2.9. *Let $\xi(A)$ be an additive set function defined on the semiring $\mathfrak{H}$. Suppose that for every $A \in \mathfrak{H}$ the moment $M_k(A)$ exists and the following two conditions hold:*

1. *The set functions $M_i(A)$ $(i = 1, \ldots, k-1;\ A \in \mathfrak{H})$ are of bounded variation.*

2. *If $B_1, B_2, \ldots$ is a sequence of sets of $\mathfrak{H}$ such that $\lim_{n\to\infty} d(B_n) = 0$, then*

$$\lim_{n\to\infty} M_i(B_n) = 0 \qquad (i = 1, \ldots, k-1).$$

In this case for every $A \in \mathfrak{H}$ we have

$$\varkappa_k(A) = \mathsf{S}_A M_k(dB). \tag{2.36}$$

PROOF. It can easily be verified that $\varkappa_k(B)$ equals the sum of $M_k(B)$ and a finite number of products of the form $M_i(B) M_j(B)$ $(1 \leq i \leq k-1,$ $1 \leq j \leq k-1)$. Hence the theorem follows by a simple argument.

We remark that the results of this chapter contain implicitly the solution of the functional equations

$$F(x, A_1 + A_2) = F(x, A_1) * F(x, A_2)$$

and

$$F\left(x, \sum_{k=1}^{\infty} A_k\right) = \mathop{*}_{k=1}^{\infty} F(x, A_k),$$

accordingly as $\xi(A)$ is only additive or also completely additive. The sets $A_1, A_2, \ldots, A_1+A_2$ and $\sum_{k=1}^{\infty} A_k$ belong to $\mathfrak{R}$ or $\mathfrak{S}$ according to the problem. The properties of the set function $\xi(A)$ used in this chapter can be formulated in terms of the distribution functions. Thus we need not at all random variables, it is quite sufficient to consider the distributions and the solutions of the above functional equations follow from the modified form of our theorems.

III. CONSIDERATIONS ON THE REALIZATIONS

§ 1. Preliminaries

In this chapter we assume that the space H and the σ-ring $\mathfrak{S}$ consisting of some subsets of H satisfy the following condition:

α_1) Every set $A \in \mathfrak{S}$ has a sequence of subdivisions $\mathfrak{z}_n = \{B_1^{(n)}, \ldots, B_{k_n}^{(n)}\}$ with the property that $\mathfrak{z}_n \sqsubset \mathfrak{z}_{n+1}$ and if $h_1 \in H, h_2 \in H, h_1 \neq h_2$, then for some positive integers i, k, N we have

$$h_1 \in B_i^{(N)}, \quad h_2 \in B_k^{(n)}, \quad i \neq k. \tag{3. 1}$$

This condition holds e. g. for countable-dimensional Euclidean spaces.

We start from a completely additive set function $\xi(A) = \xi(\omega, A)$ $(\omega \in \Omega)$ which is supposed to be defined on the σ-ring $\mathfrak{S}$. If ω is fixed, we get a number-valued set function. This will be called a *realization* of the set function ξ. The complete additivity of ξ does not imply in general that of the realizations as it was shown in [19] (Chapter III, § 1). But in this chapter we need the complete additivity of the realizations. Therefore we introduce the condition:

β_1) There exists a set $\Omega_1 \subseteq \Omega$ with $\mathbf{P}(\Omega_1) = 1$ such that for every fixed $\omega \in \Omega_1$ the number-valued set function $\xi(\omega, A)$ $(A \in \mathfrak{S})$ is completely additive.

Besides α_1) and β_1) we suppose the condition:

γ_1) The stochastic set function $\xi(A)$ $(A \in \mathfrak{S})$ is atomless.

Condition α_1) implies that if $h \in H$, then $\{h\} \in \mathfrak{S}$. In fact, every set $\{h\}$ can be obtained as a limit of a non-decreasing sequence of sets of $\mathfrak{S}$. Thus every realization can be decomposed as a sum of a continuous and a purely discontinuous part. We shall show in § 3 that under general conditions the continuous part must be identically a constant number-valued set function with probability 1. This is perhaps the most interesting result of this chapter.

§ 2. Qualitative discussion of the discontinuities

If μ is a completely additive set function defined on the σ-ring $\mathfrak{S}$ and for an $h \in H$ we have $\mu(\{h\}) \neq 0$, then we call h a *discontinuity point* relative to μ. The number $\mu(\{h\})$ will be called the *magnitude of the discontinuity.*

Let I be a one-dimensional interval with a positive distance from the point 0 and define the functions $\chi_0(I, A)$, $\chi_1(I, A)$ of the sample elements as follows: $\chi_0(I, A)$ is the number and $\chi_1(I, A)$ the sum of those discontinuities in the set A the magnitudes of which lie in the interval I. First of all we prove a theorem relative to this functions.

THEOREM 3. 1. *If $I_1, \dots, I_r$ are disjoint intervals with positive distances from 0, then $\chi_0(I_1, A), \dots, \chi_0(I_r, A)$ and similarly $\chi_1(I_1, A), \dots, \chi_1(I_r, A)$ are independent random variables.*

In the general case when $I_1, \dots, I_r$ are arbitrary disjoint intervals, the assertion remains true for $\chi_1(I_1, A), \dots, \chi_1(I_r, A)$ and holds also for $\chi_0(I_1, A), \dots,$ $\chi_0(I_r, A)$ if the realizations belonging to Ω_1 have a finite number of discontinuities.

PROOF. Here and in the sequel we shall frequently use the following remark:

There exists a sequence of subdivisions $\mathfrak{z}_n = \{B_1^{(n)}, \dots, B_{k_n}^{(n)}\}$ of the set A such that $\mathfrak{z}_n \sqsubset \mathfrak{z}_{n+1}$, Condition α_1) holds and finally the random variables

$$\xi(B_1^{(n)}), \dots, \xi(B_{k_n}^{(n)}) \qquad (n = 1, 2, \dots) \tag{3.2}$$

are infinitesimal. In fact, the proof of Theorem 3. 2 shows that the last property can be fulfilled with a sequence $\mathfrak{z}_n'$. Then, if $\mathfrak{z}_n''$ satisfies α_1), the superposition of $\mathfrak{z}_n'$ and $\mathfrak{z}_n''$ satisfies both requirements.

First we consider the case when the intervals I_k have positive distances from 0. Let us define the functions

$$f_k^{(0)}(x) = \begin{cases} 1 & \text{if } x \in I_k, \\ 0 & \text{if } x \notin I_k, \end{cases} \tag{3.3}$$

$$f_k^{(1)}(x) = \begin{cases} x & \text{if } x \in I_k, \\ 0 & \text{if } x \notin I_k \end{cases} \qquad (k = 1, \dots, r), \tag{3.4}$$

and consider the sums

$$\chi_{0s}^{(n)} = \sum_{k=1}^{k_n} f_s^{(0)}(\xi(B_k^{(n)})), \quad \chi_{1s}^{(n)} = \sum_{k=1}^{k_n} f_s^{(1)}(\xi(B_k^{(n)})) \qquad (s = 1, \dots, r). \tag{3.5}$$

Condition α_1) implies

$$\lim_{n \to \infty} \chi_{0s}^{(n)} = \chi_0(I_s, A), \quad \lim_{n \to \infty} \chi_{1s}^{(n)} = \chi_1(I_s, A), \tag{3.6}$$

hence $\chi_0(I_s, A)$, $\chi_1(I_s, A)$ $(s = 1, \dots, r)$ are random variables.

The assertion relative to the independence will be proved by the aid of Theorem 1b of [21]. Let us consider the double sequence (3. 4) of independent random variables and apply the functions $f_1^{(0)}(x), \dots, f_r^{(0)}(x)$ (and $f_1^{(1)}(x), \dots, f_r^{(1)}(x)$, resp.) to its members. Clearly

$$f_i^{(0)}(x) f_k^{(0)}(x) = 0 \quad (f_i^{(1)}(x) f_k^{(1)}(x) = 0) \quad \text{if} \quad i \neq k.$$

Now we shall verify the fulfilment of the conditions of Theorem 1b of [21]. Relation (3. 6) implies d). Condition b) follows immediately by choosing τ so that $\tau < \delta$, where δ is the minimal distance of the intervals I_s $(s = 1, \dots, r)$ from the point 0. As for Condition c), there exists a $K > 0$ such that

$$f_s^{(0)}(x) \leq K|x| \qquad (s = 1, \dots, r),$$

moreover clearly

$$f_s^{(1)}(x) \leq |x| \qquad (s = 1, \dots, r),$$

hence for every s $(1 \leq s \leq r)$ and $\varepsilon > 0$ we have

$$\sup_{1 \leq k \leq k_n} \mathbf{P}(f_s^{(0)}(\xi(B_k^{(n)})) > \varepsilon) \leq \sup_{1 \leq k \leq k_n} \mathbf{P}\left(|\xi(B_k^{(n)})| > \frac{\varepsilon}{K}\right) \to 0,$$

$$\sup_{1 \leq k \leq k_n} \mathbf{P}(|f_s^{(1)}(\xi(B_k^{(n)}))| > \varepsilon) \leq \sup_{1 \leq k \leq k_n} \mathbf{P}(|\xi(B_k^{(n)})| > \varepsilon) \to 0$$

if $n \to \infty$.

The remaining part of the theorem can be proved by the aid of the precedings by substituting for those intervals I_s which have the point 0 inside or as a limit point, intervals having a distance ε from 0 and then taking the limit $\varepsilon \to 0$. This completes the proof.

Let I be an interval with a positive distance from 0. From the proof of Theorem 3. 1 it is clear that if $A_1, \dots, A_m$ are disjoint sets of $\mathfrak{S}$, then the random variables $\chi_0(I, A_1), \dots, \chi_0(I, A_m)$ (and $\chi_1(I, A_1), \dots, \chi_1(I, A_m)$, resp.) are independent (cf. the relations (3. 5) and (3. 6)). If $A_1, A_2, \dots$ is a sequence of disjoint sets of $\mathfrak{S}$, then in view of β_1)

$$\chi_0(I, A) = \sum_{k=1}^{\infty} \chi_0(I, A_k), \tag{3. 7}$$

$$\chi_1(I, A) = \sum_{k=1}^{\infty} \chi_1(I, A_k) \tag{3. 8}$$

where $A = \sum_{k=1}^{\infty} A_k$. The relations (3. 7) and (3. 8) hold not only with probability 1 but they are satisfied in the ordinary sense if $\omega \in \Omega_1$.

Thus for fixed I, $\chi_0(I, A)$ and $\chi_1(I, A)$ are completely additive (stochastic) set functions having completely additive realizations with probability 1.

The same holds for $\chi_1(I, A)$ without any restriction on the interval I and also for $\chi_0(I, A)$ if almost all realizations have a finite number of discontinuities. We formulate the foregoing statements in the form of a theorem.

THEOREM 3. 2. *For every fixed interval* I, $\chi_1(I, A)$ $(A \in \mathfrak{S})$ *is a completely additive set function.*

$\chi_0(I, A)$ $(A \in \mathfrak{S})$ *is also a completely additive set function if* I *has a positive distance from* 0 *or almost all realizations of the set function* $\xi(A)$ $(A \in \mathfrak{S})$ *have a finite number of discontinuities.*

If $\omega \in \Omega_1$ *is fixed, then the corresponding sample function of* $\chi_1(I, A)$ *(and under the above condition that of* $\chi_0(I, A)$*) is completely additive.*

§ 3. The probability distributions of the random variables $\xi(A), \chi_0(I, A), \chi_1(I, A)$

Now we fix the set A, the interval I and determine the probability distributions of the random variables $\xi(A), \chi_i(I, A)$ $(i = 1, 2)$. Concerning them we prove three theorems and finally formulate an interesting conclusion (Theorem 3. 6). Let us consider first $\xi(A)$.

THEOREM 3. 3. *If* $A \in \mathfrak{S}$, *then the canonical form of* $\log f(t, A)$ *is given by*

$$(3.9)\quad \log f(t, A) = i\gamma(A)t + \int\limits_{(-\infty,0)} (e^{itx} - 1)\, dM(x, A) + \int\limits_{(0,\infty)} (e^{itx} - 1)\, dN(x, A)$$

where $\gamma(A), M(x, A), N(x, A)$ *have the properties formulated in Theorem* 2. 3, *but besides these also*

$$(3.10)\qquad \int\limits_{[-1,0)} |x|\, dM(x, A) + \int\limits_{(0,1]} x\, dN(x, A) < \infty .$$

PROOF. Let $\mathfrak{z}_n = \{B_1^{(n)}, \ldots, B_{k_n}^{(n)}\}$ be a sequence of subdivisions of the set A such that $\beta_1)$ is fulfilled and the random variables in the double sequence (3. 2) are infinitesimal.

On account of the complete additivity of the realizations the non-decreasing sequence of the random variables

$$\xi_n = \sum_{k=1}^{k_n} |\xi(B_k^{(n)})|$$

has a finite limit. But for every A we have

$$\xi(A) = \sum_{k=1}^{k_n} \xi(B_k^{(n)}),$$

hence by Theorem 1 of [24], (3. 9) and (3. 10) hold.

The other statements were proved in Theorem 2. 3. Q. e. d.

Now we consider the random variable $\chi_n(I, A)$ and prove that it depends on a Poisson distribution. This is expressed more precisely by

THEOREM 3. 4. *For every $A \in \mathfrak{S}$ and every interval I with a positive distance from the point 0 the random variable $\chi_n(I, A)$ depends on a Poisson distribution with the expectation*

$$(3.11)\quad M(\chi_0(I, A)) = \begin{cases} M(b, A) - M(a, A) & \text{if } I = [a, b), \\ M(b+0, A) - M(a, A) & \text{if } I = [a, b], \\ M(b, A) - M(a+0, A) & \text{if } I = (a, b), \\ M(b+0, A) - M(a+0, A) & \text{if } I = (a, b] \end{cases} \Bigg\} (b < 0),$$
$$ \begin{cases} N(b, A) - N(a, A) & \text{if } I = [a, b), \\ N(b+0, A) - N(a, A) & \text{if } I = [a, b], \\ N(b, A) - N(a+0, A) & \text{if } I = (a, b), \\ N(b+0, A) - N(a+0, A) & \text{if } I = (a, b] \end{cases} \Bigg\} (0 < a).$$

Relation (3. 11) *holds also when $a = -\infty$ or $b = \infty$. (In this case only the third, forth, fifths and seventh rows have a meaning and $M(-\infty, A) = N(+\infty, A) = 0$.)*

PROOF. We prove (3. 11) for $I = [a, b), a > 0$, supposing that a and b are points of continuity of $N(x, A)$. The case $b < 0$ can be treated similarly and the other assertions follow from these by limit processes.

Let $\mathfrak{Z}_n = \{B_1^{(n)}, \ldots, B_{k_n}^{(n)}\}$ be a sequence of subdivisions of the set A with the property that $\beta_1)$ is fulfilled, the random variables

$$\xi(B_1^{(n)}), \ldots, \xi(B_{k_n}^{(n)}) \qquad (n = 1, 2, \ldots)$$

are infinitesimal and finally[8]

$$\lim_{n \to \infty} \sum_{k=1}^{k_n} (F(b, B_k^{(n)}) - F(a, B_k^{(n)})) = N(b, A) - N(a, A).$$

This last property can be prescribed by Theorem 2. 4.

Let us define the random variables

$$(3.12)\qquad \xi'(B_k^{(n)}) = \begin{cases} 1 & \text{if } \xi(B_k^{(n)}) \in I, \\ 0 & \text{if } \xi(B_k^{(n)}) \notin I. \end{cases}$$

Since

$$\xi'(B_k^{(n)}) \leq K|\xi(B_k^{(n)})| \qquad (k = 1, \ldots, k_n;\ n = 1, 2, \ldots)$$

where K is a constant, the random variables in the double sequence

$$\xi'(B_1^{(n)}), \ldots, \xi'(B_{k_n}^{(n)}) \qquad (n = 1, 2, \ldots)$$

[8] Cf. the proof of Theorem 2. 4.

are infinitesimal. Obviously

$$\mathbf{M}(\xi'(B_k^{(n)})) = F(b, B_k^{(n)}) - F(a, B_k^{(n)}),$$

hence the characteristic function of $\xi'(B_k^{(n)})$ equals

$$1 + (F(b, B_k^{(n)}) - F(a, B_k^{(n)}))(e^{it} - 1).$$

On the other hand

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} (F(b, B_k^{(n)}) - F(a, B_k^{(n)}))^2 = 0, \tag{3.13}$$

hence for every t

$$\lim_{n\to\infty} \prod_{k=1}^{k_n} \{1 + (F(b, B_k^{(n)}) - F(a, B_k^{(n)}))(e^{it} - 1)\} =$$

$$\exp\left\{\lim_{n\to\infty} \sum_{k=1}^{k_n} (F(b, B_k^{(n)}) - F(a, B_k^{(n)}))(e^{it} - 1)\right\} \tag{3.14}$$

$$= \exp\{(N(b, A) - N(a, A))(e^{it} - 1)\}.$$

Taking into account that

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \xi'(B_k^{(n)}) = \chi_0(I, A),$$

our theorem follows.

Now we determine the charachteristic function of the random variable $\chi_1(I, A)$.

THEOREM 3. 5. *For every set $A \in \mathfrak{S}$ and every interval I we have*

$$M(e^{i\chi_1(I,A)t}) = \exp\left\{\int_{X'I} (e^{itx} - 1)\, dM(x, A) + \int_{X'I} (e^{itx} - 1)\, dN(x, A)\right\} \tag{3.15}$$

where X' denotes the real line without the point 0.

PROOF. For simplicity we suppose that I lies on the positive half-axis. The general case does not require any new arguments. We suppose even that $I = [a, b)$, and the function $N(x, A)$ is continuous at the points a, b. Clearly

$$\chi_1(I, A) = \lim_{n\to\infty} \sum_{k=0}^{n-1} \left(a + \frac{b-a}{n}k\right) \chi_0\left(\left[a + \frac{b-a}{n}k,\, a + \frac{b-a}{n}(k+1)\right), A\right), \tag{3.16}$$

hence for every t

$$M(e^{i\chi_1(I,A)t}) = \lim_{n\to\infty} \exp\left\{\sum_{k=0}^{n-1} e^{\left(a + \frac{b-a}{n}k\right)t}\left(N\left(a + \frac{b-a}{n}(k+1)\right) - N\left(a + \frac{b-a}{n}k\right)\right)\right\} =$$

$$= \exp\left\{\int_{[a,b)} (e^{itx} - 1)\, dN(x, A)\right\}.$$

For an arbitrary interval I the proof can be carried out by a limit procedure and the theorem follows.

If we denote the sum of discontinuities in the set $A \in \mathfrak{S}$ by $\zeta(A)$, then according to Theorem 3. 5

$$\text{(3. 17)} \qquad M(e^{i\zeta(A)t}) = \exp\left\{ \int_{(-\infty,0)} (e^{itx}-1)\, dM(x, A) + \int_{(0,\infty)} (e^{itx}-1)\, dN(x, A) \right\}.$$

Let $\eta(A)$ denote the difference $\xi(A)-\zeta(A)$ and prove the independence of the random variables $\zeta(A)$ and $\eta(A)$.

Define the functions

$$f_\varepsilon(x) = \begin{cases} x & \text{if } |x| \leqq \varepsilon, \\ 0 & \text{otherwise}, \end{cases} \qquad g_\varepsilon(x) = \begin{cases} 0 & \text{if } |x| \leqq \varepsilon, \\ x & \text{if } |x| > \varepsilon \end{cases} \qquad (\varepsilon > 0),$$

and choose a sequence of subdivisions $\mathfrak{Z}_n = \{B_1^{(n)}, \ldots, B_{k_n}^{(n)}\}$ of the set A so that it has the property β_1) and the random variables

$$(**) \qquad \xi(B_1^{(n)}), \ldots, \xi(B_{k_n}^{(n)}) \qquad (n=1, 2, \ldots)$$

are infinitesimal. Denoting by I_ε the set of real numbers $(-\infty, \varepsilon)+(\varepsilon, \infty)$, we get

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} f_\varepsilon(\xi(B_k^{(n)})) = \chi_1(I_\varepsilon, A),$$

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} g_\varepsilon(\xi(B_k^{(n)})) = \xi(A) - \chi_1(I_\varepsilon, A).$$

Now we apply Theorem 1b of [21] to the double sequence (**) and the functions $f_\varepsilon(x), g_\varepsilon(x)$. We have only to verify Condition c) of this theorem (cf. [21], p. 322). If $\tau < \varepsilon$, then

$$\int_{|x|<\tau} x\, d_x \mathbf{P}(g_\varepsilon(\xi(B_k^{(n)})) < x) = 0 \qquad (k=1, \ldots, k_n;\ n=1, 2, \ldots).$$

On the other hand, by Theorem 3. 8 of [19], there is a constant K_τ such that

$$\sum_{k=1}^{k_n} \left| \int_{|x|<\tau} x\, d_x\, \mathbf{P}(f_\varepsilon(\xi(B_k^{(n)})) < x) \right| = \sum_{k=1}^{k_n} \left| \int_{|x|<\tau} x\, dF(x, B_k^{(n)}) \right| \leqq K_\tau .$$

Hence and from the relation

$$\lim_{n\to\infty} \sup_{1\leqq k\leqq k_n} \left| \int_{|x|<\tau} x\, dF(x, B_k^{(n)}) \right| = 0$$

the fulfilment of Condition c) follows.

Thus $\chi_1(I_\varepsilon, A)$ and $\xi(A)-\chi_1(I_\varepsilon, A)$ are independent for every $\varepsilon>0$. By taking the limit $\varepsilon\to 0$ we conclude that $\zeta(A)$ and $\eta(A)$ are also independent.

In view of the equality

$$\xi(A)=\zeta(A)+\eta(A)$$

and of Theorem 3. 3 (the sum of the integrals on the right-hand side of (3. 9) is the characteristic function of $\zeta(A)$) we must have

$$\mathbf{P}(\eta(A)=\gamma(A))=1.$$

Hence we get the following

THEOREM 3. 6. *The completely additive set function ξ equals the sum of a completely additive set function ζ having (completely additive and) purely discontinuous realizations with probability* 1 *and a completely additive set function η the random variables of which are constant with probability* 1 (cf. (3. 17)).

If the σ-ring $\mathfrak{S}$ has a countable basis,[9] *then there exists a set $\Omega_0\subseteq\Omega_1$ with $\mathbf{P}(\Omega_0)=1$ and a number-valued completely additive set function $\gamma(A)$ $(A\in\mathfrak{S})$ such that every realization of $\xi(A)-\gamma(A)$ $(A\in\mathfrak{S})$ belonging to Ω_0 is purely discontinuous.*

PROOF. We have only to prove the second assertion. If $A_1, A_2, \ldots$ is a basis of $\mathfrak{S}$ and $\Omega^{(k)}$ is the set of those ω's for which

(3. 18) $$\eta(\omega, A_k)=\gamma(A_k),$$

then we define Ω_0 as

$$\Omega_0=\Omega_1\Omega^{(1)}\Omega^{(2)}\Omega^{(3)}\ldots.$$

Let $\omega\in\Omega_0$. In view of

$$\eta(\omega, A_k)=\gamma(A_k)\qquad (k=1, 2, \ldots)$$

we must have also

$$\eta(\omega, A)=\gamma(A)$$

for an arbitrary set $A\in\mathfrak{S}$. Since $\mathbf{P}(\Omega_0)=1$, the proof is complete.

§ 4. General characterization of the set function $\xi(A)$

Let X_ε denote the set $(-\infty, \varepsilon)+(\varepsilon, \infty)$ and consider the product spaces $X\times H$, $X_\varepsilon\times H$. We denote by $\mathfrak{R}$ and $\mathfrak{R}_\varepsilon$ the rings consisting of finite sums of sets of the type

$$Y\times A \quad\text{and}\quad Y_1\times A,$$

respectively, where $Y\subset X$ and $Y_1\subseteq X_\varepsilon$ are intervals and $A\in\mathfrak{S}$. Consider the

[9] I. e. there exists a sequence $A_1, A_2, \ldots$ of sets of $\mathfrak{S}$ such that the smallest σ-ring containing these sets is $\mathfrak{S}$.

stochastic set functions χ_1 and χ_0 defined on the σ-rings $\mathfrak{S}(\mathfrak{R})$ and $\mathfrak{S}(\mathfrak{R}_\varepsilon)$, respectively, as follows. If for an $\omega \in \Omega_1$ (Ω_1 denotes the same set as in § 3), the points of discontinuity of the realization are $h_1, h_2, \ldots$, and their magnitudes are in the same order $x_1, x_2, \ldots$, then let

$$\chi_1(B) = \sum_{(x_k, h_k) \in B} x_k \qquad (B \in \mathfrak{S}(\mathfrak{R})), \tag{3.19}$$

$$\chi_0(B) = \sum_{(x_k, h_k) \in B} 1 \qquad (B \in \mathfrak{S}(\mathfrak{R}_\varepsilon)). \tag{3.20}$$

On the set $\Omega - \Omega_1$ we may define the functions of the sample elements (3. 19)—(3. 20) arbitrarily. We shall prove

THEOREM 3. 7. *For every $B \in \mathfrak{S}(\mathfrak{R})$ and $B \in \mathfrak{S}(\mathfrak{R}_\varepsilon)$ the functions $\chi_1(B)$ and $\chi_0(B)$ are random variables, respectively. To disjoint sets $B_1, B_2, \ldots$ of $\mathfrak{S}(\mathfrak{R})$ and $\mathfrak{S}(\mathfrak{R}_\varepsilon)$ there belong independent random variables $\chi_1(B_1), \chi_1(B_2), \ldots$ and $\chi_0(B_1), \chi_0(B_2), \ldots$, respectively, and if $B = \sum_{k=1}^{\infty} B_k$, then*

$$\chi_1(B) = \sum_{k=1}^{\infty} \chi_1(B_k) \quad \textit{and} \quad \chi_0(B) = \sum_{k=1}^{\infty} \chi_0(B_k), \tag{3.21}$$

respectively.

PROOF. The fulfilment of the relations (3. 21) is obvious. Let $\mathfrak{M}$ denote the class of those sets B of $\mathfrak{S}(\mathfrak{R})$ and $\mathfrak{S}(\mathfrak{R}_\varepsilon)$ for which $\chi_1(B)$ and $\chi_0(B)$ are random variables, respectively. If $B_1, B_2, \ldots$ is a monotone sequence of sets of $\mathfrak{M}$ and $B = \lim_{n \to \infty} B_n$, then (3. 21) implies

$$\lim_{n \to \infty} \chi_1(B_n) = \chi_1(B) \quad \text{and} \quad \lim_{n \to \infty} \chi_0(B_n) = \chi_0(B),$$

respectively, for $\omega \in \Omega_1$. Hence $B \in \mathfrak{M}$ and $\mathfrak{M}$ is a monotone class of sets. But obviously $\mathfrak{R} \subset \mathfrak{M}$ and $\mathfrak{R}_\varepsilon \subset \mathfrak{M}$, hence by Theorem B of [11], p. 27 we have $\mathfrak{S}(\mathfrak{R}) = \mathfrak{M}$ and $\mathfrak{S}(\mathfrak{R}_\varepsilon) = \mathfrak{M}$, respectively.

It remains to prove the assertion relative to the independence. We show that it holds for disjoint sets belonging to $\mathfrak{R}$ and $\mathfrak{R}_\varepsilon$, respectively. Let us consider the system of sets $B_1, \ldots, B_r$ defined by

$$B_k = I_k \times A_k \tag{3.22}$$

where $A_k \in \mathfrak{S}$ ($k = 1, \ldots, r$) and $I_1, \ldots, I_r$ are intervals in the spaces X and X_ε, respectively.

The sets A_k and the intervals I_k ($k = 1, \ldots, r$) can be represented as sums of disjoint sets $C_1, \ldots, C_M$ ($C_k \in \mathfrak{S}$; $k = 1, \ldots, M$) and intervals $J_1, \ldots, J_N$,

respectively. Consider the NM sets

$$
\begin{array}{c}
J_1 \times C_1, \ldots, J_1 \times C_M, \\
\cdots\cdots\cdots\cdots\cdots \\
J_N \times C_1, \ldots, J_N \times C_M.
\end{array}
\tag{3.23}
$$

The sets $B_1, \ldots, B_r$ can be represented as sums of disjoint groups of the sets (3.23), hence it suffices to prove the independence of the random variables

$$
\begin{array}{ccc}
\chi_1(J_1 \times C_1), \ldots, \chi_1(J_1 \times C_M), & & \chi_0(J_1 \times C_1), \ldots, \chi_0(J_1 \times C_M), \\
\cdots\cdots\cdots\cdots\cdots & \text{and} & \cdots\cdots\cdots\cdots\cdots \\
\chi_1(J_N \times C_1), \ldots, \chi_1(J_N \times C_M) & & \chi_0(J_N \times C_1), \ldots, \chi_0(J_N \times C_M),
\end{array}
\tag{3.24}
$$

respectively. But according to Theorem 3.1 in every column of (3.24) the random variables are independent. On the other hand, the vectors

$$
\begin{aligned}
\vec{\chi}_1^{(i)} &= (\chi_1(J_1, C_i), \ldots, \chi_1(J_N, C_i)), \\
\vec{\chi}_0^{(i)} &= (\chi_0(J_1, C_i), \ldots, \chi_0(J_N, C_i))
\end{aligned}
\tag{3.25}
$$

are constructed by the aid of the random variables $\{\xi(C), C \in C_i \mathfrak{S}\}$ and the sets $C_1, \ldots, C_M$ are disjoint, hence the vectors (3.25) (if i runs over $1, \ldots, M$) must be independent.

Thus $\chi_1(B)$ and $\chi_0(B)$ are completely additive stochastic set functions on the rings $\mathfrak{R}$ and $\mathfrak{R}_\varepsilon$, respectively. The extension of both set functions is obviously possible. The extended set functions must coincide with $\chi_1(B)$ $(B \in \mathfrak{S}(\mathfrak{R}))$ and $\chi_0(B)$ $(B \in \mathfrak{S}(\mathfrak{R}_\varepsilon))$, respectively, hence taking into account that the extension process leads to a completely additive set function, the theorem follows.

Finally, we give an integral representation of the random variables $\xi(A)$ $(A \in \mathfrak{S})$. If $A \in \mathfrak{S}$ is a fixed set and we introduce the notation

$$\bar{\chi}_1(Y) = \chi_1(Y \times A)$$

where $Y \subseteq X_\varepsilon$ $(\varepsilon > 0)$ is a Borel set, then $\bar{\chi}_1(Y)$ is a completely additive set function defined on the σ-ring of the Borel sets of the space X_ε. Thus $\xi(A) - \gamma(A)$ equals the improper integral

$$\xi(A) - \gamma(A) = \lim_{\varepsilon \to 0} \int_{X_\varepsilon} y \bar{\chi}_1(dY). \tag{3.26}$$

For every fixed Y the random variable $\bar{\chi}_1(Y)$ depends on a Poisson distribution. Thus we can formulate:

Every completely additive set function $\xi(A)$ $(A \in \mathfrak{S})$ satisfying the requirements $\alpha_1)$, $\beta_1)$ and $\gamma_1)$ of § 1, can be represented as the limit (3.26) *of stochastic integrals* (cf. [20]) *taken relative to completely additive set functions of Poisson type.*

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DIE EINSTUFIG NICHTKOMMUTATIVEN ENDLICHEN RINGE

Von

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§ 1. Einleitung

Für ein Axiomensystem A für (algebraische) Strukturen soll eine A-Struktur eine dem Axiomensystem A genügende Struktur bedeuten. Ähnlich heißt eine A-Unterstruktur einer Struktur eine Unterstruktur dieser Struktur, die zugleich eine A-Struktur ist.

Unter einer *einstufig nichtkommutativen A-Struktur* verstehen wir jede nichtkommutative A-Struktur, deren von ihr verschiedene A-Unterstrukturen kommutativ sind. Je nachdem A das Axiomensystem für Gruppen, Ringe usw. bezeichnet, sprechen wir in diesem Sinne über die *einstufig nichtkommutativen Gruppen, Ringe* usw.

Für unendliche Strukturen ist das Problem der einstufig nichtkommutativen Strukturen äußerst schwierig. Ein triviales Beispiel für die einstufig nichtkommutativen Schiefkörper bildet der Quaternionenkörper über dem rationalen Zahlkörper. Dagegen wissen wir nicht, ob einstufig nichtkommutative unendliche Gruppen oder Ringe überhaupt existieren.

Innerhalb der endlichen Strukturen ist den einstufig nichtkommutativen Strukturen eine große Bedeutung beizumessen, denn in voller Allgemeinheit gilt die triviale aber wichtige Folgerung der Definition, daß jede nichtkommutative endliche A-Struktur mindestens eine einstufig nichtkommutative A-Struktur (als Unterstruktur) enthält. Wenn man also für ein Axiomensystem A alle einstufig nichtkommutativen endlichen A-Strukturen erforscht hat, so bietet das im gesagten Sinne auch schon eine gewisse Aufklärung über die sämtlichen nichtkommutativen endlichen A-Strukturen.

Im Gegensatz zu ihrer Wichtigkeit sind die einstufig nichtkommutativen endlichen Gruppen sehr spät erforscht worden. Wohl wurden sie nämlich schon im Jahre 1903 durch Miller—Moreno [4] untersucht,[1] doch erst im Jahre 1947 durch Rédei [5] restlos bestimmt.[2] Man siehe auch Rédei [6].

[1] Mit [ ] verweisen wir auf das Literaturverzeichnis am Ende der Arbeit.

[2] Ich war im Jahre 1924 im Besitz meiner vollständigen Resultate; die Verspätung der Publikation geschah, weil ich lange Zeit der Unvollständigkeit obenerwähnter früherer

Angewendet wurden diese Gruppen bisher bei GOLFAND [1], ITÔ [2], ITÔ—SZÉP [3], RÉDEI [7, 8]. Vgl. noch SUZUKI [11].

Der Zweck der vorliegenden Arbeit ist die Bestimmung der einstufig nichtkommutativen endlichen Ringe.

Gleich hier bemerken wir die folgenden drei Eigenschaften der einstufig nichtkommutativen Ringe, die aus der Definition trivial folgen:

Jeder solche Ring wird durch seine irgend zwei nichtvertauschbaren Elemente erzeugt.

Die nichtkommutativen homomorphen Bilder jedes solchen Ringes sind wieder einstufig nichtkommutativ.

Jeder solche Ring ist direkt unzerlegbar, also im uns allein interessierenden endlichen Fall ein p-Ring (p Primzahl), d. h. ein Ring, dessen Elemente nach der Addition eine (kommutative) p-Gruppe, mit anderen Worten einen p-Modul bilden.

Unsere Arbeit gliedert sich so. Wir schicken im § 2 einige Definitionen und durchgängige Bezeichnungen voran. Im § 3 definieren wir gewisse drei (unendliche) Klassen von Ringen R_1, R_2, R_3 und sprechen sechs Sätze 1, 2, 3, 1′, 2′, 3′ aus, von denen die ersten drei die wichtigsten Eigenschaften dieser Ringe beschreiben und die letzten drei feststellen, daß sie im wesentlichen (nämlich teils mit ihren nichtkommutativen homomorphen Bildern, teils mit den zu ihnen antiisomorphen Ringen zusammen) eben die sämtlichen einstufig nichtkommutativen endlichen Ringe sind. In den §§ 4 bis 8 erfolgt der Beweis unserer Sätze. Im § 9 machen wir einige Schlußbemerkungen verschiedener Natur.

§ 2. Einige Definitionen und durchgängige Bezeichnungen

$\{\ \}, \{\ \}^+, \{\ \}^\times, \{\ \}^*, (\ )$ bezeichnen bzw. den (die, das) durch die eingeklammerten Elemente erzeugten (erzeugte) Unterring, Untermodul, Teilhalbgruppe, Untergruppe, Ideal einer Struktur S. Stets wird aus den Zusammenhängen klar oder nötigenfalls extra gesagt, welche Struktur S wir dabei meinen. Mit ( ) bezeichnen wir auch den größten gemeinsamen Teiler.

R^+ und $R^\times$ bezeichnen den Modul bzw. die Halbgruppe (der Elemente) eines Ringes R. Unter einer *Basis* von R verstehen wir eine solche von R^+.

Untersuchungen nicht gewahr wurde. Es ist zu bemerken, daß das Problem der einstufig nichtkommutativen endlichen Gruppen im Zusammenhang mit dem der einstufig nichtnilpotenten endlichen Gruppen (s. § 9) im Jahre 1924 auch SCHMIDT [10] betrachtet und in ihrer Theorie eine wichtige Vereinfachung erzielt hat, jedoch lag ihm dabei an der genauen Erforschung der erstgenannten Gruppen sichtbar nicht.

K^* bezeichnet die Gruppe (der von 0 verschiedenen Elemente) eines Körpers K.

$O(S)$ bezeichnet die *Ordnung* (d. h. die Anzahl der Elemente) *einer endlichen Struktur* S.

$o(\alpha)$ bezeichnet die (multiplikative) *Ordnung*, d. h. die Anzahl der verschiedenen Potenzen $\alpha, \alpha^2, \ldots$ *des Elementes* α einer endlichen Struktur, in der (unter anderem) die Multiplikation der Elemente definiert ist. Wenn also α insbesondere ein nilpotentes Element (einer Struktur mit Nullelement) ist, so bedeutet $o(\alpha)$ den *Nilpotenzgrad* von α, d. h. die kleinste natürliche Zahl a mit $\alpha^a = 0$. Wörtlich ebenso definieren wir den Nilpotenzgrad eines nilpotenten Ringes R und bezeichnen ihn ähnlich mit $o(R)$.

$o^+(\alpha)$ bezeichnet die *additive Ordnung* von α, d. h. das additive Analogon von $o(\alpha)$, wobei α immer ein Element eines (endlichen) Moduls oder Ringes sein wird, weshalb $o^+(\alpha)$ stets die kleinste natürliche Zahl a mit $a\alpha = 0$ bedeutet.

$o(\alpha(\mathrm{mod}\,\mathfrak{a})) = o(\alpha + \mathfrak{a})$ bezeichnet die (multiplikative) Ordnung von $\alpha \bmod \mathfrak{a}$, d. h. die (multiplikative) Ordnung der Restklasse $\alpha(\mathrm{mod}\,\mathfrak{a}) (= \alpha + \mathfrak{a})$, wobei α ein Element und $\mathfrak{a}$ ein Ideal eines Ringes ist.

$[\alpha, \beta] (= \alpha\beta - \beta\alpha)$ bezeichnet den *Kommutator* der Elemente α, β eines Ringes.

$\mathfrak{J}$ bezeichnet den Ring der ganzen Zahlen.

$\mathfrak{J}[x]$ bezeichnet den Polynomring von x über $\mathfrak{J}$.

$\mathfrak{J}[x]_0 (= x\mathfrak{J}[x])$ bezeichnet den Unterring von $\mathfrak{J}[x]$ bestehend aus den durch x teilbaren Polynomen (aus $\mathfrak{J}[x]$).

$\mathfrak{R}_{x,y}$ bezeichnet den durch die Elemente x, y (als freie Erzeugende) erzeugten *freien Ring*. Dabei fassen wir $\mathfrak{J}[x]_0$ als einen Unterring von $\mathfrak{R}_{x,y}$ auf.

$\mathfrak{J} + R$ bezeichnet den bekannten (gemeinsamen) Erweiterungsring von $\mathfrak{J}$ und R, dessen Elemente eindeutig durch

$$a + \alpha \qquad (a \in \mathfrak{J},\ \alpha \in R)$$

angegeben sind und nach den Regeln

$$(a+\alpha)+(b+\beta) = (a+b)+(\alpha+\beta),\quad (a+\alpha)(b+\beta) = ab+(a\beta+b\alpha+\alpha\beta)$$

addiert und multipliziert werden; dabei ist R ein beliebiger Ring, der mit $\mathfrak{J}$ kein gemeinsames Element außer 0 hat. Immer, wenn nötig, fassen wir R als einen Ring versehen mit dem zweiseitigen Operatorenbereich $\mathfrak{J} + R$ auf, indem wir die Operationen durch

$$(a+\alpha)\beta = a\beta + \alpha\beta,\quad \beta(a+\alpha) = a\beta + \beta\alpha \quad (a \in \mathfrak{J};\ \alpha, \beta \in R)$$

erklären.[3] Da $\mathfrak{J}[x] = \mathfrak{J} + \mathfrak{J}[x]_0$ gilt, so folgt, daß die

$$f(\alpha)\beta, \quad \beta f(\alpha) \qquad\qquad (f(x) \in \mathfrak{J}[x];\ \alpha, \beta \in \mathsf{R})$$

einen Sinn haben, und zwar Elemente von R sind (obwohl selbst $f(\alpha)$ nicht in R zu liegen braucht). Ist dabei

$$f(x) = c + g(x) \qquad\qquad (c \in \mathfrak{J},\ g(x) \in \mathfrak{J}[x]_0),$$

so gelten nach obigem

$$f(\alpha)\beta = c\beta + g(\alpha)\beta, \quad \beta f(\alpha) = c\beta + \beta g(\alpha).$$

$\mathfrak{A} <\!\!=\!\!> \mathfrak{B}$ bezeichnet die Äquivalenz der Aussagen $\mathfrak{A}, \mathfrak{B}$.

$\mathsf{S} \approx \mathsf{S}'$ und $\mathsf{S} \sim \mathsf{S}'$ bezeichnen die Isomorphie bzw. Homomorphie der Strukturen S, S'.

Ist $\mathfrak{M}$ eine Menge von Strukturen $\mathsf{S}_1, \mathsf{S}_2, \ldots$, denen der Reihe nach gewisse Systeme $\iota_1, \iota_2, \ldots$ von Dingen eindeutig zugeordnet sind derart, daß die Regel

$$\iota_i = \iota_j <\!\!=\!\!> \mathsf{S}_i \approx \mathsf{S}_j$$

gilt, so sagen wir, daß ι_i (innerhalb $\mathfrak{M}$) ein *vollständiges Invariantensystem* von S_i ist ($i = 1, 2, \ldots$).

p bezeichnet eine Primzahl.

Die Elemente eines endlichen Primkörpers bezeichnen wir üblicherweise mit den ganzen rationalen Zahlen, die freilich nur mod p in Betracht kommen, wobei p die Charakteristik des Körpers ist.

Die *einseitigen Nullteiler* eines Ringes bedeuten diejenigen Links- und Rechtsnullteiler des Ringes, die keine Rechts- bzw. Linksnullteiler sind.

Hauptpolynom bedeutet ein Polynom in einer Unbestimmten über einem Ring mit Einselement, dessen Anfangskoeffizient das Einselement ist.

Ohne besondere Erklärung zu verwendende kleine lateinische Buchstaben bezeichnen ganze Zahlen.

§ 3. Aufzählung der einstufig nichtkommutativen endlichen Ringe

Wir definieren die folgenden Ringe $\mathsf{R}_1, \mathsf{R}_2, \mathsf{R}_3$:

Erstens werde ein Ring

(1) $$\mathsf{R}_1 = \{\varrho, \sigma\}$$

[3] Nach einer mündlichen Mitteilung von Herrn A. KERTÉSZ hat er vor kurzem allgemeiner im Zusammenhang mit beliebigen Operatormoduln eine der obigen ähnliche Konstruktion ausgearbeit, nach der dann jeder Operatormodul sehr vorteilhaft in einen unitären Modul verwandelt wird. — *Bei der Korrektur:* Man vgl. auch die inzwischen erschienene Arbeit R. E. JOHNSON, Structure theory of faithful rings II. Restricted rings, *Trans. Amer. Math. Soc.*, **84** (1957), 523—544.

durch die Gleichungen

$$p^m \varrho = 0, \quad p^n \sigma = 0, \quad \varrho^r = 0, \quad \sigma^s = 0, \tag{2}$$

$$\varrho^2 \sigma = \varrho \sigma \varrho = \sigma \varrho^2, \quad \varrho \sigma^2 = \sigma \varrho \sigma = \sigma^2 \varrho, \tag{3}$$

$$p \sigma \varrho = p \varrho \sigma \tag{4}$$

definiert, wobei m, n, r, s natürliche Zahlen sind mit den Eigenschaften:

$$1 \leqq m \leqq n; \; r \geqq 2, \; s \geqq 2; \quad \text{wenn} \quad m = n, \quad \text{so} \quad r \leqq s. \tag{5}$$

Zweitens werde ein Ring

$$\mathrm{R}_2 = \{\varrho, \sigma\} \tag{6}$$

durch die Gleichungen

$$p^m \varrho = 0, \quad p\sigma = 0, \quad \varrho F(\varrho) = 0, \quad \sigma^2 = 0, \quad F(\varrho)\sigma = 0, \quad \sigma \varrho = \varrho^{p^{n q^{e-1}}} \sigma \tag{7}$$

definiert, wobei q eine Primzahl ist (die auch gleich p sein mag), $m, n (n < q), e$ natürliche Zahlen sind, $F(x) (\in \mathfrak{J}[x])$ ein mod p irreduzibles Hauptpolynom q^e-ten Grades ist.

Drittens werde ein Ring

$$\mathrm{R}_3 = \{\varrho, \sigma\} \tag{8}$$

durch die Gleichungen

$$p^m \varrho = 0, \quad p\sigma = 0, \quad \varrho^2 = \varrho, \quad \sigma^2 = 0, \quad \varrho\sigma = \sigma, \quad \sigma\varrho = 0 \tag{9}$$

definiert, wobei m eine natürliche Zahl ist.

Solche Ringe $\mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3$ gibt es, wie wir sehen werden, je unendlich viele, ferner sind sie offenbar lauter endliche p-Ringe. Um ihre Abhängigkeit von den jeweils genannten Parametern zum Ausdruck zu bringen, hätten wir uns der ausführlicheren Bezeichnungen

$$\mathrm{R}_1 = \mathrm{R}_1(p, m, n, r, s), \quad \mathrm{R}_2 = \mathrm{R}_2(p, m, q, e, n), \quad \mathrm{R}_3 = \mathrm{R}_3(p, m)$$

bedienen können, jedoch nehmen wir davon Abstand, bemerken aber schon im voraus, daß die hier eingeklammerten Parameter jedesmal ein vollständiges Invariantensystem bilden (s. Sätze 1, 2, 3). Scheinbar hängt R_2 auch noch vom Polynom $F(x)$ ab, jedoch werden wir sehen (Satz 2), daß die (bei gegebenen p, m, q, e, n) zu den verschiedenen $F(x)$ gehörenden R_2 isomorph sind.

SATZ 1. *Jeder Ring* R_1 *ist einstufig nichtkommutativ, hat das vollständige Invariantensystem*

$$p, m, n, r, s, \tag{10}$$

ist nilpotent vom Nilpotenzgrad

$$o(\mathrm{R}_1) = r + s - 1, \tag{11}$$

und endlich von der Ordnung

$$O(\mathrm{R}_1) = p^{1 + m(r-1)s + n(s-1)}. \tag{12}$$

Die rs Elemente

$$\varrho\sigma-\sigma\varrho, \quad \varrho^i\sigma^j \quad (0 \leqq i \leqq r-1;\ 0 \leqq j \leqq s-1;\ i+j \neq 0) \tag{13}$$

bilden eine Basis von R_1 *mit*

$$\begin{gathered} o^+(\varrho\sigma-\sigma\varrho)=p, \\ o^+(\varrho^i\sigma^j)=p^m \ (1 \leqq i \leqq r-1;\ 0 \leqq j \leqq s-1), \quad o^+(\sigma^j)=p^n \ (1 \leqq j \leqq s-1). \end{gathered} \tag{14}$$

Für das Produkt zweier durch diese Basis ausgedrückter Elemente gilt

$$\begin{gathered} \Big(a(\varrho\sigma-\sigma\varrho)+\sum_{i,j} a_{ij}\varrho^i\sigma^j\Big)\Big(b(\varrho\sigma-\sigma\varrho)+\sum_{k,l} b_{ij}\varrho^k\sigma^l\Big)= \\ =-a_{01}b_{10}(\varrho\sigma-\sigma\varrho)+\sum_{i,j,k,l} a_{ij}b_{kl}\varrho^{i+k}\sigma^{j+l}. \end{gathered} \tag{15}$$

SATZ 2. *Jeder Ring* R_2 *ist einstufig nichtkommutativ, hat das vollständige Invariantensystem*

$$p, m, q, e, n, \tag{16}$$

ist nichtnilpotent, enthält keine einseitigen Nullteiler und ist endlich von der Ordnung

$$O(\mathrm{R}_2)=p^{(m+1)q^e}. \tag{17}$$

Die Elemente von R_2 *lassen sich in der Form*

$$a(\varrho)+b(\varrho)\sigma \qquad (a(x)\in\mathfrak{J}[x]_0,\ b(x)\in\mathfrak{J}[x]) \tag{18}$$

schreiben. Für zwei in dieser Form angenommene Elemente

$$\mu=a(\varrho)+b(\varrho)\sigma, \quad \nu=c(\varrho)+d(\varrho)\sigma \quad (a(x), c(x)\in\mathfrak{J}[x]_0;\ b(x), d(x)\in\mathfrak{J}[x]) \tag{19}$$

gelten die Regeln[4, 5]

$$\mu=\nu \iff a(x)\equiv c(x) \pmod{p^m, F(x)}, \ b(x)\equiv d(x) \pmod{p, F(x)}, \tag{20}$$

$$\mu\nu=a(\varrho)c(\varrho)+\big(a(\varrho)d(\varrho)+b(\varrho)c(\varrho)^{p^{nq^{e-1}}}\big)\sigma. \tag{21}$$

Die Elemente

$$\varrho^i \ (i=1,\dots,q^e), \quad \varrho^j\sigma \ (j=0,\dots,q^e-1) \tag{22}$$

bilden eine Basis von R_2 *mit*

$$o^+(\varrho^i)=p^m, \quad o^+(\varrho^j\sigma)=p. \tag{23}$$

[4] Offenbar ist (20) äquivalent mit dem (oft zu verwendenden) Spezialfall:
$a(\varrho)+b(\varrho)\sigma=0 \iff a(x)\equiv 0 \pmod{p^m, F(x)}, \quad b(x)\equiv 0 \pmod{p, F(x)}$.

[5] Da

$$f(x)^{p^i}\equiv f(x^{p^i}) \pmod{p} \qquad (f(x)\in\mathfrak{J}[x];\ i=1,2,\dots)$$

ist, so folgt aus (20) die Regel

$$f(\varrho)^{p^i}\sigma=f(\varrho^{p^i})\sigma.$$

Entsprechend läßt sich also (21) in einer zweiten Form schreiben, die manchmal vorteilhaft ist.

Satz 3. *Jeder Ring* R_3 *ist einstufig nichtkommutativ, hat das vollständige Invariantensystem*

$$p, m, \tag{24}$$

ist nichtnilpotent, enthält σ *als einseitigen Nullteiler und ist endlich von der Ordnung*

$$O(\mathsf{R}_3) = p^{m+1}. \tag{25}$$

Die Elemente ϱ, σ *bilden eine Basis von* R_3 *mit*

$$o^+(\varrho) = p^m, \quad o^+(\sigma) = p. \tag{26}$$

Für das Produkt zweier durch diese Basis ausgedrückter Elemente gilt

$$(a\varrho + b\sigma)(c\varrho + d\sigma) = ac\varrho + ad\sigma. \tag{27}$$

Unsere Hauptresultate sind enthalten in den folgenden drei Sätzen, die insgesamt alle einstufig nichtkommutativen endlichen Ringe aufzählen:

Satz 1′. *Die nilpotenten einstufig nichtkommutativen endlichen Ringe stimmen mit den Ringen* R_1 *und ihren nichtkommutativen homomorphen Bildern überein.*

Satz 2′. *Die nichtnilpotenten einstufig nichtkommutativen endlichen Ringe ohne einseitige Nullteiler stimmen mit den Ringen* R_2 *überein.*

Satz 3′. *Die nichtnilpotenten einstufig nichtkommutativen endlichen Ringe mit einseitigen Nullteilern stimmen mit den Ringen* R_3 *und den zu diesen antiisomorphen Ringen überein.*

§ 4. Beweis vom Satz 1

Aus Satz 1 wollen wir zuerst die Behauptungen über (12), (13), (14) beweisen.

Nach der Definition bei (1) bis (5) ist R_1 isomorph zum Faktorring $\mathfrak{R}_{xy}/\mathfrak{a}$ des freien Ringes $\mathfrak{R}_{xy}$ nach dem Ideal

$$\mathfrak{a} = (p^m x, p^n y, x^r, y^s, x^2y - yx^2, x^2y - xyx, xy^2 - y^2x, xy^2 - yxy, p(xy - yx)). \tag{28}$$

Da ferner die Endlichkeit von R_1 schon aus (1), (2) folgt, so gilt

$$O(\mathsf{R}_1) = O(\mathfrak{R}_{xy}/\mathfrak{a}). \tag{29}$$

Wegen (28) und $m \leqq n$ repräsentieren die Elemente

$$a\,yx + \sum_{i,j} a_{ij} x^i y^j \qquad \left(\begin{array}{l} a = 0, \ldots, p-1, \\ a_{ij} = 0, \ldots, p^m - 1;\ i > 0, \\ a_{0j} = 0, \ldots, p^n - 1 \end{array}\right) \tag{30}$$

alle Restklassen mod $\mathfrak{a}$ von $\mathfrak{R}_{r,s}$, wobei man über die i, j mit

$$0 \leq i \leq r-1; \quad 0 \leq j \leq s-1; \quad i+j \neq 0$$

zu summieren hat.

Wir zeigen, daß dabei jede Klasse nur einmal repräsentiert wird. Es genügt zu beweisen, daß (30) nur dann in $\mathfrak{a}$ liegt, wenn a und alle a_{ij} gleich 0 sind.

Zu diesem Zweck müssen wir etwas länger ausholen. Es durchlaufe P alle Elemente der Halbgruppe $\{x, y\}^{*} (\subset \mathfrak{R}_{r,s}^{*})$. Die sämtlichen P bilden eine Basis von $\mathfrak{R}_{r,s}$. Es sei

(31) $$\sum c_P P$$

die Basisdarstellung eines Elementes von $\mathfrak{R}_{r,s}$, wobei man über alle P zu summieren hat und die c_P ganze Zahlen sind, von denen nur endlich viele nicht verschwinden. Die $c_P P (\neq 0)$ nennen wir die Glieder von (31). Enthält dabei P genau k Faktoren x und l Faktoren y, so sagen wir, daß dieses Glied vom Grad k bzw. l bezüglich x bzw. y ist, ferner nennen wir $k+l$ schlechthin den Grad dieses Gliedes. Für ein Element $(\neq 0)$ von $\mathfrak{R}_{r,s}$ verstehen wir unter dem Grad dieses Elementes das Maximum der Grade seiner Glieder. Sind die letzteren Grade untereinander gleich, so nennen wir das betrachtete Element homogen. Offenbar läßt sich jedes Element von $\mathfrak{R}_{r,s}$ eindeutig in der Form

$$H_1 + H_2 + \cdots$$

schreiben, wobei H_t gleich 0 oder homogen vom Grad t ist und nur endlich viele H_t von 0 verschieden sind. Die $H_1, H_2, \ldots$ nennen wir die homogenen Bestandteile des betrachteten Elementes von $\mathfrak{R}_{r,s}$. Eine triviale Folgerung ist, daß ein Element von $\mathfrak{R}_{r,s}$ dann und nur dann in ein durch homogene Elemente erzeugbares Ideal gehört, wenn seine homogenen Bestandteile in diesem Ideal liegen.

Nunmehr nehmen wir an, daß (30) in $\mathfrak{a}$ gehört. Wie gesagt, haben wir zu zeigen, daß a und alle a_{ij} gleich 0 sind.

Wegen (28) und $r \geq 2, s \geq 2$ gilt $\mathfrak{a} \subset (px, py, x^2, xy, y^2)$. Aus der Annahme folgt somit

$$a\,yx + a_{01}x + a_{10}y \in (px, py, x^2, xy, y^2).$$

Da die rechte Seite homogene Erzeugende hat, so folgt aus obiger Bemerkung:

$$a\,yx \in (px, py, x^2, xy, y^2).$$

Offenbar folgt hieraus $a\,yx \in (px, py)$, also $p|a$. Da aber a nur der Werte $0, \ldots, p-1$ fähig ist, so haben wir

(32) $$a = 0.$$

Es ist noch zu zeigen, daß auch alle a_{ij} gleich 0 sind. Wir nehmen an, daß das falsch ist, und bezeichnen mit u $(0 \leq u \leq s-1)$ die größte ganze

Zahl von der Eigenschaft, daß die sämtlichen a_{ij} mit $j < u$ gleich 0 sind. Nach der Annahme ist (30) ein Element von $\mathfrak{a}$. Man multipliziere (30) von rechts mit y^{s-1-u}. Auch dieses Produkt liegt in $\mathfrak{a}$, also ist wegen (32)

$$\sum_{i,j} a_{ij} x^i y^{s-1-u+j} \in \mathfrak{a}.$$

Hier fallen nach der Definition von u die Glieder mit $j < u$ heraus. Da ferner nach (28) $y^s \in \mathfrak{a}$ ist, so lassen sich auch die Glieder mit $s-1-u+j \geq s$, d. h. mit $j > u$ streichen. Hiernach bleiben nur die Glieder mit $j = u$ übrig:

$$\sum_{i} a_{iu} x^i y^{s-1} \in \mathfrak{a}.$$

(Je nachdem $u = 0$ oder $u > 0$ ist, hat man hier über $i = 1, \ldots, p-1$ bzw. $i = 0, \ldots, p-1$ zu summieren.) Die Glieder der linken Seite sind von verschiedenem Grade. Andererseits hat $\mathfrak{a}$ nach (28) lauter homogene Erzeugende, weshalb nach obiger Bemerkung alle Glieder in $\mathfrak{a}$ liegen müssen. Da nun nach der Definition von u mindestens ein a_{iu} von 0 verschieden ist, so gilt hierfür

$$a_{iu} x^i y^{s-1} \in \mathfrak{a}.$$

Wegen (28) gilt noch mehr

$$a_{iu} x^i y^{s-1} \in (p^m x, p^n y, x^r, y^s, xy-yx),$$

denn die gestrichenen Erzeugenden von $\mathfrak{a}$ schon im Hauptideal $(xy-yx)$ liegen. Wegen $i < r$, $s-1 < s$ folgt sogar

$$a_{iu} x^i y^{s-1} \in (p^m x, p^n y, xy-yx). \tag{33}$$

Da $\mathfrak{R}_{r,s}$ ein freier Ring ist, so darf in (33) die Ersetzung $y \to x$ ausgeführt werden:

$$a_{iu} x^{i+s-1} \in (p^m x, p^n x),$$

wobei rechts nunmehr ein Ideal von $\mathfrak{J}[x]_0$ steht. Wegen $m \leq n$ folgt hieraus

$$p^m \mid a_{iu}.$$

Da aber a_{iu} im Fall $i > 0$ wegen (30) und $a_{iu} \neq 0$ nur der Werte $1, \ldots, p^m - 1$ fähig ist, so folgt notwendig $i = 0$ $(u > 0)$.

In diesem Fall lautet (33) so:

$$a_{0u} y^{s-1} \in (p^m x, p^n y, xy-yx).$$

Nach der Ersetzung $x \to 0$, $y \to x$ ergibt sich

$$a_{0u} x^{s-1} \in (p^n x),$$

also

$$p^n \mid a_{0u}.$$

Da aber a_{0u} wegen (30) und $a_{0u} \neq 0$ nur der Werte $1, \ldots, p^n - 1$ fähig ist, so haben wir mit diesem Widerspruch bewiesen, daß die Elemente (30) eben die sämtlichen verschiedenen Restklassen mod $\mathfrak{a}$ von $\mathfrak{R}_{r,s}$ repräsentieren.

Da die Anzahl dieser Repräsentanten (30) gleich

$$p \cdot (p^m)^{(r-1)s} \cdot (p^n)^{s-1}$$

ist, so folgt wegen (29) die Richtigkeit von (12).

Aus (1), $(2_{3,4})$, (3) folgt sofort, daß die Elemente (13) Erzeugende des Moduls R_1^+ sind. Andererseits folgt aus $(2_{1,2})$, (4), (5_1), daß für diese Elemente die Aussagen (14) mit „$\leqq$" statt „$=$" richtig sind. Da aber das Produkt der rechten Seiten aller Gleichungen (14) gleich der rechten Seite der eben bewiesenen Gleichung (12) ist, so ersieht man hieraus auf einmal, daß die Elemente (13) sogar eine Basis von R_1 bilden und alle Gleichungen (14) richtig sind.

Wird $\varrho\sigma - \sigma\varrho$ von links oder rechts mit ϱ oder σ multipliziert, so entsteht nach (3) stets 0. Hieraus und wieder aus (3) folgt die Richtigkeit von (15).

Um (11) zu beweisen beachte man, daß wegen (3) der Modul $(\mathsf{R}_1^k)^+$ des Ringes R_1^k im Fall $k \geqq 3$ durch die Elemente

$$\varrho^i \sigma^j \qquad (i + j = k)$$

erzeugt wird. Diese verschwinden aber wegen $(2_{3,4})$ stets, wenn $k \geqq r + s - 1$ ist. Da nach $(5_{2,3})$ auch $r + s - 1 \geqq 3$ gilt, so folgt

$$\mathsf{R}_1^{r+s-1} = 0.$$

Andererseits enthält R_1^{r+s-2} das Element $\varrho^{r-1}\sigma^{s-1}$. Dieses ist nach (13) ein Basiselement, also jedenfalls von 0 verschieden. Hieraus folgt

$$\mathsf{R}_1^{r+s-2} \neq 0.$$

Somit haben wir (11) bewiesen.

Zum übrigen Beweis brauchen wir den folgenden

Hilfssatz 1. *Wenn für einen Ring* R *und einen Unterring* S *von ihm*

(34) $$\mathsf{R} = \mathsf{R}^2 + \mathsf{S}$$

gilt, so gilt sogar

(35) $$\mathsf{R} = \mathsf{R}^k + \mathsf{S} \qquad (k = 2, 3, \ldots).$$

Insbesondere für einen nilpotenten Ring R *kann also* (34) *nur mit* $\mathsf{S} = \mathsf{R}$ *erfüllt werden.*

Denn nehmen wir (35) für ein k ($\geqq 2$) an. Hieraus und aus (34) folgt

$$\mathsf{R} = (\mathsf{R}^2 + \mathsf{S})^k + \mathsf{S} \subset \mathsf{R}^{k+1} + \mathsf{S}^k + \mathsf{S} = \mathsf{R}^{k+1} + \mathsf{S}.$$

Dies hat die Richtigkeit von (35) für $k + 1$ statt k, also auch allgemein zur Folge.

Für die späteren Zwecke beweisen wir noch

Hilfssatz 2. *In einem minimalen Erzeugendensystem* $\omega_1, \omega_2, \ldots$ *eines nilpotenten Ringes* R *läßt sich kein* ω_i *durch ein* $\omega_i\omega_j$ *oder* $\omega_j\omega_i$ *ersetzen* ($j = i$ *oder* $j \neq i$).

Denn bezeichne S den (echten) Unterring $\{\omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots\}$ von R. Ist die Behauptung falsch, d. h. ist

$$\mathrm{R} = \{\mathrm{S}, \omega_i\omega_j\} \quad \text{oder} \quad \mathrm{R} = \{\mathrm{S}, \omega_j\omega_i\},$$

so enthält der Unterring $\mathrm{R}^2 + \mathrm{S}$ von R ein Erzeugendensystem von R, woraus (34) folgt. Nach Hilfssatz 1 müßte also $\mathrm{S} = \mathrm{R}$ sein. Dieser Widerspruch beweist Hilfssatz 2.

Nunmehr wollen wir beweisen, daß R_1 einstufig nichtkommutativ ist. Da $\varrho\sigma - \sigma\varrho$ nach (13) ein Basiselement von R_1, also jedenfalls von 0 verschieden ist, so ist R_1 nicht kommutativ. Deshalb genügt es zu beweisen, daß R_1 durch seine irgend zwei nichtvertauschbaren Elemente α, β erzeugt wird.

Wir drücken α, β mit Hilfe der Basis (14) aus und bezeichnen die dabei auftretenden Koeffizienten mit a, a_{ij} bzw. b, b_{ij}, genau so wie in den beiden Faktoren auf der linken Seite von (15). Aus (15) folgt dann

$$[\alpha, \beta] = \alpha\beta - \beta\alpha = \begin{vmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{vmatrix} [\varrho\sigma - \sigma\varrho].$$

Wegen (14_1) und der Annahme ist die Determinante auf der rechten Seite durch p nicht teilbar. Wenn man also das Elementepaar α, β einer passenden ganzzahligen homogen linearen (mod p regulären) Substitution unterwirft und für das so erhaltene „neue" Elementepaar die alten Bezeichnungen beibehält, so gilt für die „neuen" Koeffizienten die Matrizenkongruenz

$$\begin{pmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{p^n}.$$

Da die ausgeführte Substitution mod p^n invertierbar ist, so genügt es zu zeigen, daß R_1 durch die „neuen" ϱ, σ erzeugt wird.

Die eben ausgeführte Reduktion hat (wegen $\mathrm{R}_1 = \{\varrho, \sigma\}$)

$$\varrho \equiv \alpha \pmod{\mathrm{R}_1^2}, \quad \sigma \equiv \beta \pmod{\mathrm{R}_1^2}$$

zur Folge. Nach der letzten Aussage vom Hilfssatz 1 angewendet auf den (nilpotenten) Ring $\mathrm{R} = \mathrm{R}_1 = \{\varrho, \sigma\}$ und seinen Unterring $\mathrm{S} = \{\alpha, \beta\}$ gilt also $\mathrm{S} = \mathrm{R}_1$. Das besagt, daß α, β tatsächlich Erzeugende von R_1 sind, d. h. R_1 einstufig nichtkommutativ ist.

Endlich haben wir von Satz 1 nur noch zu beweisen, daß die Zahlen (10) ein vollständiges Invariantensystem von R_1 bilden. Da R_1 durch diese

Zahlen bis auf Isomorphie bestimmt ist, so genügt es zu zeigen, daß sie lauter Invarianten von R_1 sind.

Da (13) eine Basis aus R_1 ist, so bilden die rechten Seiten der Gleichungen (14) ein System von Invarianten von R_1. Dieses System besteht aus den folgenden Elementen:

$$p, (r-1)s\text{-mal } p^m, \; s-1\text{-mal } p^n.$$

Im Fall $m < n$ folgt hieraus sofort, daß selbst p, m, n, r, s Invarianten von R_1 sind.

Im übriggebliebenen Fall $m = n$ folgt auf diesem Wege nur, daß p, m, rs Invarianten von R_1 sind. Da aber in diesem Fall vor allem $r \leqq s$ gilt, ferner nach (11) stets auch $r+s$ eine Invariante von R_1 ist, so kommt man auch jetzt zum vorigen Resultat. Somit haben wir Satz 1 bewiesen.

§ 5. Beweis vom Satz 2

Im hier folgenden Beweis vom Satz 2 wird das mit Hilfe des bei (7) eingeführten $\bmod p$ irreduziblen Hauptpolynoms $F(x)$ gebildete Primideal

$$(36) \qquad (p, F(x))$$

von $\mathfrak{J}[x]$ eine führende Rolle spielen. Da $F(x)$ vom Grad $q^r (>1)$ ist, so ist der Faktorring

$$(37) \qquad \mathrm{K} = \mathfrak{J}[x]/(p, F(x))$$

ein (endlicher) Körper mit

$$(38) \qquad O(\mathrm{K}) = p^{q^r} (>p).$$

Wegen (37) gibt es in K ein erzeugendes Element $\varkappa$ mit

$$(39) \qquad \mathrm{K} = \{\varkappa\}, \quad F(\varkappa) = 0.$$

(Für $\varkappa$ läßt sich wegen (37) die Restklasse $x + (p, F(x))$ nehmen; jedoch wird K in unseren Betrachtungen nur als abstrakter Körper in Frage kommen, und dann wird es bloß auf die Existenz von $\varkappa$ ankommen.)

Wir benötigen auch den (endlichen) Faktorring

$$(40) \qquad \mathrm{A} = \mathfrak{J}[x]_0/(p^m x, F(x)x).$$

Hierfür gilt offenbar

$$(41) \qquad O(\mathrm{A}) = p^{m q^r}.$$

Vor allem wollen wir einen Ring R effektiv konstruieren, von dem wir dann zeigen werden, daß er zum Ring R_2 isomorph ist. Nachdem dies geschehen ist, werden wir die im Satz 2 behaupteten Eigenschaften von R_2 ausweisen können.

Zu unserem Zweck bezeichnen wir mit $\mathfrak{R}$ zunächst die Menge aller (geordneten) Paare

$$\langle \alpha, \beta \rangle \qquad (\alpha \in \mathfrak{A},\ \beta \in \mathfrak{K}) \tag{42}$$

und verabreden uns unter

$$\langle a(x), b(x) \rangle' \qquad (a(x) \in \mathfrak{J}[x]_0,\ b(x) \in \mathfrak{J}[x]) \tag{43}$$

das aus den Restklassen

$$\alpha = a(x) + (p^m x, F(x)x), \quad \beta = b(x) + (p, F(x)) \tag{44}$$

gebildete Element (42) von $\mathfrak{R}$ zu verstehen. Es ist klar, daß umgekehrt alle Elemente von $\mathfrak{R}$ sich in der Form (43) schreiben lassen. Dann definieren wir in $\mathfrak{R}$ die Addition und Multiplikation durch

$$\langle a(x), b(x) \rangle' + \langle c(x), d(x) \rangle' = \langle a(x)+c(x),\ b(x)+d(x) \rangle', \tag{45}$$

$$\langle a(x), b(x) \rangle' \langle c(x), d(x) \rangle' = \langle a(x)c(x),\ a(x)d(x) + b(x)c(x)^{p^{nq^{r-1}}} \rangle', \tag{46}$$

wobei $a(x), c(x)$ und $b(x), d(x)$ in $\mathfrak{J}[x]_0$ bzw. $\mathfrak{J}[x]$ liegende Polynome sind. Es ist wegen (37) und (40) klar, daß beide Verknüpfungen (45), (46) eindeutig sind. Wir zeigen zunächst, daß $\mathfrak{R}$ ein Ring ist.

Wegen (45) bildet $\mathfrak{R}$ offenbar einen Modul. Um die Assoziativität der Multiplikation auszuweisen, bezeichnen wir mit A und B den ersten bzw. zweiten Faktor auf der linken Seite von (46), ferner nehmen wir ein drittes Element $C = \langle f(x), g(x) \rangle'$ von $\mathfrak{R}$ $(f(x) \in \mathfrak{J}[x]_0,\ g(x) \in \mathfrak{J}[x])$. Der Kürze halber setzen wir

$$P = p^{nq^{r-1}} \tag{47}$$

und $a(x) = a, \ldots, g(x) = g$. Dann gelten nach (46):

$$AB = \langle ac,\ ad + bc^P \rangle', \quad BC = \langle cf,\ cg + df^P \rangle',$$

$$(AB)C = \langle acf,\ acg + adf^P + bc^P f^P \rangle' = A(BC).$$

Das beweist die gesagte Assoziativität. Da stets

$$u(x)^{p^i} + v(x)^{p^i} \equiv (u(x) + v(x))^{p^i} \pmod p \qquad (u(x), v(x) \in \mathfrak{J}[x];\ i = 1, 2, \ldots)$$

gilt, so ersieht man aus (45), (46) die Distributivität von $\mathfrak{R}$ sofort. Hiernach ist $\mathfrak{R}$ tatsächlich ein Ring.

Wir zeigen, daß $\mathfrak{R}$ durch die zwei Elemente

$$\varrho' = \langle x, 0 \rangle', \quad \sigma' = \langle 0, 1 \rangle' \tag{48}$$

erzeugt wird:

$$\mathfrak{R} = \{\varrho', \sigma'\}. \tag{49}$$

Es genügt zu beweisen, daß das beliebige Element (43) von $\mathfrak{R}$ in der rechten Seite von (49) enthalten ist. Wir zeigen sogar

$$a(\varrho') + b(\varrho')\sigma' = \langle a(x), b(x) \rangle' \qquad (a(x) \in \mathfrak{J}[x]_0,\ b(x) \in \mathfrak{J}[x]). \tag{50}$$

Wenn nämlich $b(x)$ insbesondere in $\mathfrak{J}[x]_0$ gehört, so ist die rechte Seite von (50) nach (45), (46) gleich

$$\langle a(x), 0\rangle' + \langle b(x), 0\rangle' \langle 0, 1\rangle'.$$

Wieder nach (45), (46) und nach (48_1) gilt stets

$$\langle f(x), 0\rangle' = f(\langle x, 0\rangle') = f(\varrho') \qquad (f(x) \in \mathfrak{J}[x]_0).$$

Hiernach und nach (48_2) geht der vorige Ausdruck in die linke Seite von (50) über. Somit ist (50) in diesem Fall richtig. Im allgemeinen Fall läßt sich

$$b(x) = k + b_0(x) \qquad (k \in \mathfrak{J},\ b_0(x) \in \mathfrak{J}[x]_0)$$

setzen. Nach dem vorangeschickten Spezialfall von (50) und nach (45), (48_2) gilt

$$a(\varrho') + b(\varrho')\sigma' = a(\varrho') + k\sigma' + b_0(\varrho')\sigma' = \langle a(x), b_0(x)\rangle' + k\langle 0, 1\rangle' = \langle a(x), b(x)\rangle'.$$

Das beendet den Beweis von (50).

Wir beweisen die Homomorphie

$$\mathrm{R}_2 \sim \mathrm{R}. \tag{51}$$

Da R_2 der durch (6) und (7) definierte Ring ist und (49) gilt, so genügt es zu beweisen, daß die Gleichungen (7) auch mit ϱ', σ' (statt ϱ, σ) erfüllt sind. Das ist in der Tat der Fall, da nach (50) und der Definition von (43) (teils auch wegen (45), (46), (47), (48)),

$$\begin{gathered}
p^m \varrho' = \langle p^m x, 0\rangle' = 0,\\
p\sigma' = \langle 0, p\rangle' = 0,\\
F(\varrho')\,\varrho' = \langle F(x)\,x, 0\rangle' = 0,\\
\sigma'^2 = \langle 0, 1\rangle'^2 = \langle 0, 0\rangle' = 0,\\
F(\varrho')\,\sigma' = \langle 0, F(x)\rangle' = 0,\\
\sigma'\varrho' = \langle 0, 1\rangle' \langle x, 0\rangle' = \langle 0, x^p\rangle' = \langle x^p, 0\rangle' \langle 0, 1\rangle' = \varrho'^p \sigma'
\end{gathered}$$

gelten. Somit ist (51) bewiesen.

Andererseits ersieht man aus (7) sofort, daß alle Elemente von R_2 sich in der Form (18) schreiben lassen. Indem wir ferner beachten, daß die in (20) figurierenden Polynome $a(x), c(x)$ in $\mathfrak{J}[x]_0$ liegen, d. h. durch x teilbar sind, so sehen wir aus ($7_{1,2,3,4}$), daß aus der rechten Seite von (20) die linke Seite folgt. Wegen (37), (40) ergibt sich also

$$O(\mathrm{R}_2) \leqq O(\mathrm{A})\, O(\mathrm{K}). \tag{52}$$

Nun gilt aber nach (42)

$$O(\mathrm{R}) = O(\mathrm{A})\, O(\mathrm{K}).$$

Hieraus und aus (52) folgt in der Tat, daß die Homomorphie (51) auch als

Isomorphie

(53) $$(\mathsf{R}_2) \approx \mathsf{R}$$

gelten und (52) mit „=“ bestehen muß:

(54) $$O(\mathsf{R}_2) = O(\mathsf{A})\,O(\mathsf{K}).$$

Diese Gleichheit hat, wie wir nunmehr es obigem Beweis der Ungleichung (52) entnehmen, auch die Richtigkeit der Regel (20) zur Folge.

Aus (38), (41), (54) folgt (17).

Die Regel (21) folgt aus $(7_{2,4,6})$.

Aus der Regel (20) folgt, daß (18) die sämtlichen verschiedenen Elemente von R_2 liefert, wenn $a(x)$ die Polynome aus $\mathfrak{Z}[x]_0$ vom (formalen) Grad q^e mit Koeffizienten $0, \ldots, p^m-1$ und $b(x)$ die Polynome aus $\mathfrak{Z}[x]$ vom (formalen) Grad q^e-1 mit Koeffizienten $0, \ldots, p-1$ durchläuft. Das beweist die Behauptungen über (22), (23). Es ist nur noch übrig die vor (17) ausgesprochenen Behauptungen vom Satz 2 zu beweisen.

Da $F(x) \bmod p$ irreduzibel und nicht linear ist, so enthält das Ideal $(p^m, F(x))$ keine Potenz x^i $(i = 1, 2, \ldots)$. Dies hat wegen der Regel (20) zur Folge, daß das Element ϱ von R_2 nicht nilpotent ist. Das beweist, daß selbst R nicht nilpotent ist.

Wir zeigen, daß R_2 keine einseitigen Nullteiler enthält. Statt dessen zeigen wir sogar, daß für irgend zwei Elemente μ, ν von R_2 aus der Annahme $\mu\nu = 0$ stets $\nu\mu = 0$ folgt. Zu diesem Zweck nehmen wir μ, ν in der Form (19) an. Die Bedingung $\mu\nu = 0$ ist dann nach (21) und (20) gleichbedeutend mit

(55) $$a(x)c(x) \equiv 0 \pmod{p^m, F(x)}, \quad a(x)d(x) + b(x)c(x)^h \equiv 0 \pmod{p, F(x)},$$

wobei wieder die kurze Bezeichnung (47) verwendet wurde. Ist (55_1) erfüllt, so liegt $a(x)c(x)$ noch mehr im Primideal $(p, F(x))$, woraus das gleiche für $a(x)$ oder $c(x)$ folgt. Also läßt sich (55_2) durch $a(x)d(x), b(x)c(x) \in (p, F(x))$ ersetzen. Da hiernach (55) gegen die Vertauschung von μ mit ν invariant ist, so sind die beiden Aussagen $\mu\nu = 0$, $\nu\mu = 0$ äquivalent, was wir zeigen wollten.

Jetzt nehmen wir die Behauptung an die Reihe, daß R_2 einstufig nichtkommutativ ist.

Hiervon zeigen wir zuerst, daß R_2 nicht kommutativ ist. Da $F(x) \bmod p$ irreduzibel und vom Grad q^e ist, so gilt die bekannte Regel

(56) $$x^{p^k} \equiv x \pmod{p, F(x)} \iff q^e \mid k.$$

Da nun nach (7_6)

$$\sigma\varrho - \varrho\sigma = (\varrho^{p^{h q^{e-1}}} - \varrho)\,\sigma$$

gilt und n wegen $0 < n < q$ durch q nicht teilbar ist, so folgt aus den Regeln (20), (56), daß $\sigma\varrho - \varrho\sigma \neq 0$ ist. Somit ist R_2 tatsächlich nicht kommutativ.

Hiernach genügt es noch zu zeigen, daß R_2 durch seine irgend zwei nichtvertauschbaren Elemente erzeugt wird. Zu diesem Zweck betrachten wir zwei Elemente

(57) $\mu = a(\varrho) + b(\varrho)\,\sigma,\ \ \nu = c(\varrho) + d(\varrho)\,\sigma \quad (a(x), c(x) \in \mathfrak{J}[x]_0;\ b(x), d(x) \in \mathfrak{J}[x])$

(genau so wie in (19)) und nehmen $\mu\nu = \nu\mu$ an, was nach der Regel (21) mit

(58) $$(a(\varrho)\,d(\varrho) + b(\varrho)\,c(\varrho)^P)\,\sigma \neq (a(\varrho)^P\,d(\varrho) + b(\varrho)\,c(\varrho))\,\sigma \qquad (P = p^{nq^{e-1}})$$

gleichbedeutend ist. Wir haben

(59) $$\mathsf{R}_2 = \{\mu, \nu\}$$

auszuweisen. Das wird nach einigen ziemlich mühsamen „Reduktionsschritten" ermöglicht.

Wegen (58) ist mindestens das eine der Elemente

$$(a(\varrho)^P - a(\varrho))\,\sigma, \quad (c(\varrho)^P - c(\varrho))\,\sigma$$

von 0 verschieden. Da beide Ausdrücke bei Vertauschung von μ mit ν ineinander übergehen, so dürfen wir

(60) $$(a(\varrho)^P - a(\varrho))\,\sigma \neq 0$$

voraussetzen. Dies besagt nach der Regel (20)

$$a(x)^P \not\equiv a(x) \pmod{p, F(x)}.$$

Wegen der Bedeutung von P gilt dann noch mehr

$$a(x)^{pq^{e-1}} \not\equiv a(x) \pmod{p, F(x)}.$$

Dies ist wegen (39) gleichbedeutend mit

$$a(\varkappa)^{pq^{e-1}} \neq a(\varkappa).$$

Hieraus folgt nach (38) und der Galoisschen Theorie, daß $a(\varkappa)$ ein erzeugendes Element von K ist:

(61) $$\mathsf{K} = \{a(\varkappa)\}.$$

Wir zeigen, daß es ein Polynom $f(x)$ mit

(62) $$f(a(x)) \equiv c(x) \pmod{p, F(x)}, \quad f(x) \in \mathfrak{J}[x]_0$$

gibt (und bemerken der späteren halber, daß ähnliches für jedes Polynom aus $\mathfrak{J}[x]$ statt $c(x)$ gilt, wie wir das aus dem Beweis sehen werden).

Wegen (61) gibt es nämlich ein Polynom $f(x)$ ($\in \mathfrak{J}[x]_0$) mit $f(a(\varkappa)) = c(\varkappa)$. Dies ist wegen (39) gleichbedeutend mit (62).

Als erster Reduktionsschritt wollen wir zeigen, daß man im Beweis der Behauptung (59) sich auf den Fall

(63) $$c(x) \equiv 0 \pmod{p, F(x)}$$

beschränken darf.

Um das zu zeigen, betrachten wir das Element

(64) $$\nu_1 = \nu - f(\mu)$$

von R_2, das wir ((57$_2$) ähnlich) auch in der Form

(65) $$\nu_1 = c_1(\varrho) + d_1(\varrho)\,\sigma \qquad (c_1(x) \in \mathfrak{J}[x]_0,\ d_1(x) \in \mathfrak{J}[x])$$

ansetzen. Da nach (64) $\{\mu, \nu_1\} = \{\mu, \nu\}$ gilt, also die Behauptung (59) mit $\mathsf{R} = \{\mu, \nu_1\}$ gleichbedeutend ist, so genügt es, wenn wir zeigen, daß das Paar μ, ν_1 den über μ, ν gemachten Annahmen genügt und dabei (63) für c_1 statt c erfüllt ist.

Da wegen (64) $[\mu, \nu_1] = [\mu, \nu]$ ist, so folgt aus obiger Annahme $\mu\nu \neq \nu\mu$ die entsprechende Ungleichung $\mu\nu_1 \neq \nu_1\mu$. Deswegen bleibt (58) für ν_1 statt ν ebenfalls erhalten.

Da $a(x)$ nach (57$_1$) von ν_1 gar nicht abhängt, so erleiden (60), (61) beim Übergehen von ν auf ν_1 gar keine Änderung.

Um endlich die Erfülltheit von (63) für c_1 statt c zu zeigen, berücksichtigen wir, daß aus (57$_1$) die Kongruenz (in R_2)

$$f(\mu) \equiv f(a(\varrho)) \pmod{\sigma},$$

ferner aus (62) (wegen $x \mid a(x), f(x), c(x)$) die Kongruenz

$$f(a(\varrho)) \equiv c(\varrho) \pmod{p\varrho, F(\varrho)\varrho}$$

folgt. Wegen (57$_2$) und (64) gilt also

$$\nu_1 \equiv 0 \pmod{p\varrho, F(\varrho)\varrho, \sigma}.$$

Hiernach und nach (7$_{4,6}$), (65) ist

$$c_1(\varrho) = p\,u(\varrho) + F(\varrho)\,v(\varrho) + w(\varrho)\,\sigma$$

mit $u(x), v(x)$ aus $\mathfrak{J}[x]_0$ und $w(x)$ aus $\mathfrak{J}[x]$. Wegen (7$_{2,3,5}$) ist also

$$c_1(\varrho)\,\sigma = 0.$$

Dies ist nach der Regel (20) gleichbedeutend mit

$$c_1(x) \equiv 0 \pmod{p, F(x)},$$

weshalb wir uns in der Tat auf den Fall (63) beschränken dürfen.

Wegen (7$_{2,5}$), (63) und $x \mid c(x)$ ist $c(\varrho)\sigma = 0$. Hiernach folgt aus (58)

$$d(\varrho)\,\sigma \neq 0.$$

Dies ist nach der Regel (20) gleichbedeutend mit

(66) $$d(x) \not\equiv 0 \pmod{p, F(x)}.$$

Als eine weitere Reduktion zeigen wir, daß wir (über (66) hinaus) sogar

(67) $$d(x) = 1$$

annehmen dürfen.

Zu diesem Zweck setzen wir

(68) $$\sigma_1 = d(\varrho)\sigma.$$

Wegen (66) gibt es ein Polynom $d_1(x)\,(\in \mathfrak{F}[x])$ mit

$$d_1(x)\,d(x) \equiv 1 \pmod{p, F(x)}.$$

Nach ($7_{2,5}$) ist dann

$$d_1(\varrho)\,d(\varrho)\sigma = \sigma.$$

Nach (68) gilt also

(69) $$\sigma = d_1(\varrho)\,\sigma_1.$$

Wegen (6), (68), (69) ist $\mathrm{R}_2 = \{\varrho, \sigma\} = \{\varrho, \sigma_1\}$. Auch ist es klar, daß wegen (68) die Bedingungen ($7_{2,4,5,6}$) für σ_1 statt σ erfüllt sind. Da ferner sich (57) nach (68), (69) als

$$\mu = a(\varrho) + b(\varrho)\,d_1(\varrho)\,\sigma_1, \quad \nu = c(\varrho) + \sigma_1$$

schreiben läßt, so bedeutet das (nach dem Vergleich mit (57)), daß man sich tatsächlich auf den Fall (67) beschränken darf. Entsprechend verwandelt sich also (57) (mit unveränderten $a(x), c(x)$ aber mit neuem $b(x)$) in

(70) $$\mu = a(\varrho) + b(\varrho)\,\sigma, \quad \nu = c(\varrho) + \sigma.$$

Als letzter Reduktionsschritt zeigen wir, daß man auch noch

(71) $$b(x) = 0$$

annehmen kann.

Zu diesem Zweck setzen wir

(72) $$\mu_1 = \mu - g(\mu)\,\nu$$

mit einem bald näher zu bestimmenden Polynom $g(x)$ aus $\mathfrak{F}[x]_0$. Da aus (72) (bei beliebigem $g(x)$)

$$\{\mu, \nu\} = \{\mu_1, \nu\}$$

folgt, so wird es genügen, wenn wir unsere Behauptung (59) für μ_1 statt μ beweisen werden.

Um $g(x)$ passend zu wählen, formen wir (72) zunächst um. Nach (70_1) ist

$$g(\mu) = g(a(\varrho) + b(\varrho)\,\sigma).$$

Hieraus folgt nach ($7_{4,6}$)

(73) $$g(\mu) = g(a(\varrho)) + u(\varrho)\,\sigma$$

mit einem Polynom $u(x)$ aus $\mathfrak{F}[x]$. Aus ($7_{2,5}$), (63) folgt $c(\varrho)\,\sigma = 0, \sigma c(\varrho) = 0$. Also gilt nach (21), ($70_2$) und (73)

$$g(\mu)\,\nu = g(a(\varrho))\,c(\varrho) + g(a(\varrho))\,\sigma.$$

Hieraus und aus (70_1), (72) entsteht

(74) $$\mu_1 = a(\varrho) - g(a(\varrho))\,c(\varrho) + (b(\varrho) - g(a(\varrho)))\,\sigma.$$

Wir wenden nunmehr (62) mit $b(x)$ statt $c(x)$ an. Es folgt, daß man $g(x)$ gemäß

$$g(a(x)) \equiv b(x) \pmod{p, F(x)}$$

wählen kann. Nach $(7_{2,3})$ ist dann $g(a(\varrho))\,\sigma = b(\varrho)\,\sigma$. Bei dieser Wahl von $g(x)$ geht (74) in

(75) $$\mu_1 = a_1(\varrho)$$

über mit einem Polynom $a_1(x)$ aus $\mathfrak{J}[x]_0$, wofür nach (63)

(76) $$a_1(x) \equiv a(x) \pmod{p, F(x)}$$

gilt. Der Vergleich von (75) mit (70_1) zeigt, daß man sich im Beweis von (59) tatsächlich auf den Fall (71) beschränken kann. Da ferner aus (76) wegen $(7_{2,3})$ $a_1(\varrho)\sigma = a(\varrho)\sigma$ folgt, so bleiben (60), (61) beim Übergehen von μ auf μ_1 erhalten.

Somit haben wir den Fall (70) weiter auf den Fall

(77) $$\mu = a(\varrho), \quad \nu = c(\varrho) + \sigma$$

reduziert, wobei (60), (61), (63) unverändert gelten. Für diese μ, ν werden wir aber (59) direkt beweisen können. (Freilich folgt hieraus wegen der Nichtkommutativität von $\mathfrak{R}_2$, daß unsere ursprüngliche Annahme $\mu\nu \neq \nu\mu$ auch für (77) gilt. Diese Ungleichung $\mu\nu \neq \nu\mu$ folgt übrigens auch aus (20), (21), (60), (77). Jedoch wird $\mu\nu \neq \nu\mu$ im folgenden Beweis von (59) gar nicht benötigt.)

Auf Grund von (62) (angewendet mit x statt $c(x)$) bestimmen wir ein Polynom $h(x)$ mit

(78) $$x \equiv h(a(x)) \pmod{p, F(x)}, \quad h(x) \in \mathfrak{J}[x]_0$$

und wollen beweisen, daß es sogar für jedes $l (\geqq 1)$ ein Polynom $k(x)$ mit

(79) $$x \equiv k(a(x)) \pmod{p^l, F(x)}, \quad k(x) \in \mathfrak{J}[x]_0$$

gibt. Für $l = 1$ stimmt das wegen (78). Wir nehmen (79) für ein festes Paar $l, k(x)$ an und wollen einen Induktionsschluß machen.

Da $x \mid a(x), k(x)$ gelten, so gilt auch $x \mid k(a(x))$. Hieraus und aus (79) folgt

(80) $$x \equiv k(a(x)) + p^l\,u(x) \pmod{F(x)}$$

mit einem Polynom $u(x)$ aus $\mathfrak{J}[x]_0$. Hiernach ist

$$u(x) \equiv u(k(a(x)) + p^l\,u(x)) \pmod{F(x)},$$

also (wegen $l \geqq 1$)

$$u(x) \equiv u(k(a(x))) \pmod{p, F(x)}.$$

Dies und (80) ergeben

$$x \equiv k(a(x)) + p^l\, u(k(a(x))) \pmod{p^{l+1}, F(x)}.$$

Dies beweist die Behauptung über (79) für $l+1$ statt l, also schließlich allgemein.

Insbesondere betrachte man die Kongruenz (79) für den Fall $l=m$. Diese ist nach der Regel (20) gleichbedeutend mit $\varrho = k(a(\varrho))$. Wegen (77_1) geht diese Gleichung in $\varrho = k(\mu)$ über. Hieraus und aus (77_2) folgt noch $\sigma = \nu - c(k(\mu))$. Da hiernach beide Erzeugenden ϱ, σ in $\{\mu, \nu\}$ gehören, so ist hiermit (59) bewiesen. Das besagt, daß R_2 tatsächlich einstufig nichtkommutativ ist.

Aus Satz 2 ist nur noch die einzige Behauptung zu beweisen, daß die fünf Zahlen (16) ein vollständiges Invariantensystem von R_2 bilden. Zuerst zeigen wir, daß sie lauter Invarianten von R_2 sind.

Vor allem sind die in (23) angegebenen Ordnungen der Basiselemente (22) (bis auf die Reihenfolge) invariant durch R_2 bestimmt. Diese Ordnungen sind:

$$q^e\text{-mal } p^m \quad \text{und} \quad q^e\text{-mal } p.$$

Hieraus folgt, daß p, m, q, e Invarianten von R_2 sind.

Etwas mühsam wird der Beweis für die Invarianz von n. In Anbetracht von (6) und (7) genügt es folgendes zu beweisen: Sind ϱ', σ' zwei Elemente von R_2 und t eine natürliche Zahl mit

(81) $$\mathsf{R}_2 = \{\varrho', \sigma'\},$$

(82) $$\sigma'^2 = 0,$$

(83) $$\sigma' \varrho' = \varrho'^{p^t} \sigma',$$

(84) $$0 < t < q^e,$$

so gilt

(85) $$t = nq^{e-1}.$$

Zum Beweis setzen wir an:

(86) $$\varrho' = a(\varrho) + b(\varrho)\,\sigma, \quad \sigma' = c(\varrho) + d(\varrho)\,\sigma \quad (a(x), c(x) \in \mathfrak{F}[x]_0;\ b(x), d(x) \in \mathfrak{F}[x]).$$

Wegen (21) gilt

$$\sigma'^2 = c(\varrho)^2 + d_1(\varrho)\,\sigma$$

mit einem Polynom $d_1(x)$ aus $\mathfrak{F}[x]$. Wegen (82) folgt hieraus $c(\varrho)^2 = 0$, also nach der Regel (20)

$$c(x)^2 \equiv 0 \pmod{p^m, F(x)}.$$

Da $(p, F(x))$ ein Primideal ist, so ergibt sich aus letzterem $c(x) \equiv 0 \pmod{p, F(x)}$, d. h. nach (20)

(87) $$c(\varrho)\,\sigma = 0.$$

Wegen (81) ist $[\varrho', \sigma'] \neq 0$. Dies bedeutet nach (86) das Bestehen von (58), wofür wir jetzt wegen (87) einfach

(88) $$(a(\varrho)^P - a(\varrho))\, d(\varrho)\, \sigma \neq 0 \qquad (P = p^{nq^{e-1}})$$

schreiben dürfen.

Wegen (21) und (86_1) gilt

$$\varrho'^{p^t} = a(\varrho)^{p^t} + b_1(\varrho)\,\sigma$$

mit einem Polynom $b_1(x)$ aus $\mathfrak{J}[x]$. Da andererseits R_2 keine einseitigen Nullteiler hat, so ist nach (87) $\sigma c(\varrho) = 0$. Folglich ist wegen (7_1) und (86_2)

$$\varrho'^{p^t} \sigma' = a(\varrho)^{p^t} c(\varrho) + a(\varrho)^{p^t} d(\varrho)\, \sigma.$$

Ähnlich aber leichter entsteht aus (21) und (86)

$$\sigma' \varrho' = c(\varrho)\, a(\varrho) + d(\varrho)\, a(\varrho)^P \sigma.$$

Folglich gilt nach (83) insbesondere

(89) $$(a(\varrho)^{p^t} - a(\varrho)^P)\, d(\varrho)\, \sigma = 0.$$

Aus (88) sieht man, daß (60) auch jetzt gilt. Wie wir gesehen haben, folgt hieraus auch (61):

(90) $$\mathsf{K} = \{a(\varkappa)\}.$$

Aus (88) folgt noch $d(\varrho)\,\sigma \neq 0$. Wegen (89) ergibt sich also aus der Regel (20) (in Anbetracht der Primeigenschaft des Ideals $(p, F(x))$):

$$a(x)^{p^t} - a(x)^P \equiv 0 \pmod{p, F(x)}.$$

Dies ist wegen (39) gleichbedeutend mit

$$a(\varkappa)^{p^t} = a(\varkappa)^P.$$

Da $a(\varkappa)$ nach (90) ein Erzeugendes von K ist, so ergibt sich hieraus nach (38) (s. die Bedeutung von P bei (88)):

$$t \equiv n q^{e-1} \pmod{q^e}.$$

Dies und (84) besagen (85). Somit ist bewiesen, daß die Zahlen (16) Invarianten von R_2 sind.

Umgekehrt, nach (6) und (7) wird R_2 durch die Zahlen (16) und durch das Polynom $F(x)$ eindeutig bestimmt. Wenn wir also zeigen, daß dabei R_2 von der speziellen Wahl von $F(x)$ (bis auf Isomorphie) unabhängig ist, so werden wir unsere Behauptung bewiesen haben.

Zu diesem Zweck nehmen wir ein weiteres mod p irreduzibles Polynom $F_1(x)\,(\in \mathfrak{J}[x])$ vom Grad q^e. Es genügt in R_2 die Existenz eines Elementes ϱ_1 auszuweisen derart, daß

(91) $$\mathsf{R}_2 = \{\varrho_1, \sigma\}$$

gilt und die Gleichungen $(7_{1,3,5,6})$ mit ϱ_1, $F_1(x)$ statt ϱ, $F(x)$ erfüllt sind.

Da $F(x), F_1(x) \bmod p$ irreduzibel und von gleichem Grad sind, so gibt es ein Polynom $f(x)$ ($\in \mathfrak{J}[x]$) mit

$$(92)\qquad F_1(f(x)) \equiv 0 \pmod{p, F(x)}.$$

Dabei läßt sich $f(x)$ sogar aus $\mathfrak{J}[x]_0$ nehmen. Wir zeigen, daß für jedes $k(\geqq 1)$ sich ein Polynom $g(x)$ ($\in \mathfrak{J}[x]_0$) mit

$$(93)\qquad F_1(g(x)) \equiv 0 \pmod{p^k, F(x)}$$

angeben läßt.

Das stimmt nämlich nach (92) für $k=1$. Wir nehmen (93) für ein festes k an und zeigen die Existenz eines Polynoms $d(x)$ ($\in \mathfrak{J}[x]_0$) mit

$$(94)\qquad F_1(g(x)+p^k d(x)) \equiv 0 \pmod{p^{k+1}, F(x)},$$

wodurch die Behauptung über (93) allgemein bewiesen wird.

Wegen $k \geqq 1$ ist (94) gleichbedeutend mit

$$(95)\qquad F_1(g(x))+p^k F_1'(g(x))\, d(x) \equiv 0 \pmod{p^{k+1}, F(x)},$$

wobei F_1' den Differentialquotienten von F_1 bezeichnet. Hierfür gilt

$$(96)\qquad F_1'(g(x)) \not\equiv 0 \pmod{p, F(x)},$$

da andernfalls $F_1(x)$ wegen (92) die mehrfache Nullstelle $g(x) \bmod (p, F(x))$ hätte, was aber wegen der $\bmod p$ Irreduzibilität von $F_1(x)$ unmöglich ist. Aus (93) und (96) folgt, daß (95) eine passende Lösung $d(x)$ hat. Somit ist die Behauptung über (93) bewiesen.

Aus dem Fall $k=m$ von (93) folgt noch mehr

$$F_1(g(x))\, g(x) \equiv 0 \pmod{p^m, F(x)}.$$

Da die linke Seite durch x teilbar ist, so folgt hieraus nach der Regel (20)

$$(97)\qquad F_1(g(\varrho))\, g(\varrho)=0.$$

Da ferner nach (93)

$$(98)\qquad F_1(g(x)) \equiv 0 \pmod{p, F(x)}$$

gilt, so folgt ebenfalls nach (20)

$$(99)\qquad F_1(g(\varrho))\, \sigma=0.$$

Nach (21) ist

$$(100)\qquad \sigma g(\varrho)=g(\varrho)^p \sigma.$$

Aus (7_1) folgt auch

$$(101)\qquad p^m g(\varrho)=0.$$

Endlich zeigen wir

$$(102)\qquad R_2=\{g(\varrho), \sigma\}.$$

Wenn dies falsch ist, so gilt $[g(\varrho), \sigma]=0$, weil R_2 einstufig nichtkommutativ ist. Wegen (100) gilt also

$$(g(\varrho)^p-g(\varrho))\, \sigma=0.$$

Dies besagt nach (20):

$$g(x)^p - g(x) \equiv 0 \pmod{p, F(x)}.$$

Hieraus und aus (39), (98) folgen

$$g(z)^p = g(z), \quad F_1(g(z)) = 0, \quad K = \{g(z)\}.$$

Das erste und dritte hiervon stehen mit (38) in einem Widerspruch, womit wir (102) bewiesen haben.

Die Gleichungen (97), (99), (100), (101), (102) besagen eben, daß $\varrho_1 = g(\varrho)$ allen bei (91) gestellten Anforderungen genügt. Das beendet den Beweis vom Satz 2.

§ 6. Beweis vom Satz 3

Um Satz 3 zu beweisen stellen wir fest, daß wegen (8), (9)

(103) $$R_3 \approx \mathfrak{R}_{xy}/\mathfrak{a}$$

gilt, wobei

(104) $$\mathfrak{a} = (p^m x, py, x^2 - x, y^2, xy - y, yx)$$

ist.

Es ist klar, daß die Elemente

(105) $$ax + by \quad (a = 0, \ldots, p^m - 1;\ b = 0, \ldots, p - 1)$$

mod $\mathfrak{a}$ alle Restklassen von $\mathfrak{R}_{xy}$ repräsentieren. Um zu beweisen, daß dabei jede Klasse nur einmal repräsentiert wird, genügt es zu zeigen, daß (105) nur dann in $\mathfrak{a}$ liegt, wenn $a = b = 0$ ist.

Zu diesem Zweck nehmen wir an, daß eins der Elemente (105) in $\mathfrak{a}$ liegt:

(106) $$ax + by \in \mathfrak{a}.$$

Bei der Ersetzung $x \to 1, y \to 0$ entsteht hieraus

$$a \in (p^m),$$

wobei rechts ein Ideal von $\mathfrak{J}$ gemeint wird. Dies ergibt $a = 0$. Ferner gilt dann nach (106)

(107) $$by \in \mathfrak{a}.$$

Wir machen hier die Ersetzung $x \to E_{11}, y \to E_{12}$, wobei

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Matrizen über dem Primkörper von der Charakteristik p sind. So entsteht aus (107) $bE_{12} = 0$, also $b = 0$. Somit haben wir bewiesen, daß durch (105) jede Restklasse mod $\mathfrak{a}$ von $\mathfrak{R}_{xy}$ genau einmal repräsentiert wird.

Da die Anzahl dieser Repräsentanten (105) gleich p^{m+1} ist, so folgt aus (103) die Behauptung (25).

Indem wir die Restklassen $x+a, y+b$ mit ϱ bzw. σ bezeichnen, so geht die Isomorphie (103) (wegen (8), (9)) in die Gleichheit $R_3 = \Re_1/\mathfrak{a}$ über. Aus (105) entstehen also alle verschiedenen Elemente von R_3 in der Form

$$a\varrho+b\sigma \quad (a=0,\ldots,p^m-1;\, b=0,\ldots,p-1).$$

Dies besagt, daß ϱ, σ eine Basis von R_3 bilden und (26) besteht.

Die Regel (27) folgt unmittelbar aus (9).

Da nach (9_3) $\varrho^2=\varrho\ (\neq 0)$ ist, so ist R_3 nicht nilpotent.

Wegen $(9_{5,6})$ enthält R_3 einseitige Nullteiler.

Um zu zeigen, daß R_3 einstufig nichtkommutativ ist, betrachten wir irgend zwei nichtvertauschbare Elemente

$$\alpha=a\varrho+b\sigma,\quad \beta=c\varrho+d\sigma \quad (a,b,c,d\in\mathfrak{J})$$

von ihm. Zu zeigen ist, daß R_3 durch α, β erzeugt wird.

Nach (27) ist

$$[\alpha,\beta]=(ad-bc)\sigma\neq 0.$$

Hieraus und aus (26_2) folgt $p\nmid ad-bc$. Dies und (26) ergeben

$$\{\alpha,\beta\}^+=\{\varrho,\sigma\}^+.$$

Da die rechte Seite gleich R_3^+ ist, so folgt hieraus $R_3=\{\alpha,\beta\}$. Somit ist bewiesen, daß R_3 einstufig nichtkommutativ ist.

Wir sehen aus (8) und (9), daß R_3 durch p und m eindeutig bestimmt ist. Da ferner diese Zahlen wegen (25) Invarianten von R_3 sind, so bilden sie ein vollständiges Invariantensystem von R_3. Das beendet den Beweis vom Satz 3.

§ 7. Beweis vom Satz 1′

Da nach dem schon bewiesenen Satz 1 jeder Ring R_1 nilpotent und einstufig nichtkommutativ ist, so haben wir zum vollständigen Beweis nur folgendes zu zeigen: Ist R ein nilpotenter einstufig nichtkommutativer endlicher Ring, so ist R das homomorphe Bild eines Ringes R_1.

Wie oben bemerkt, ist R notwendig ein p-Ring, ferner gibt es in ihm zwei Elemente ϱ, σ mit

(108) $$R=\{\varrho,\sigma\},\quad \text{d. h.}\quad [\varrho,\sigma]\neq 0.$$

Wir setzen

(109) $$o^+(\varrho)=p^m,\quad o^+(\sigma)=p^n,\quad o(\varrho)=r,\quad o(\sigma)=s,$$

wobei also notwendig $m\geqq 1$, $n\geqq 1$, $r\geqq 2$, $s\geqq 2$ gelten.

Unter allen möglichen Paaren ϱ, σ wählen wir ein solches, wofür das (geordnete) Quadrupel m, n, r, s lexikographisch minimal ausfällt, d. h. erstens m, zweitens n, drittens r und viertens s minimal ist. Hieraus folgt das Bestehen von (5), da man sonst mit dem Vertauschen von ϱ und σ zum Ziele kommt.

Da ϱ, σ nach (108) ein minimales Erzeugendensystem von R ist, so folgt aus Hilfssatz 2, daß keins der Elemente $\varrho^2, \varrho\sigma, \sigma\varrho, \sigma^2$ mit ϱ oder σ zusammen den Ring R erzeugt. Also sind $\varrho^2, \varrho\sigma, \sigma\varrho, \sigma^2$ sowohl mit ϱ als auch mit σ vertauschbar:

$$\varrho^2\sigma = \varrho\sigma\varrho = \sigma\varrho^2, \quad \varrho\sigma^2 = \sigma\varrho\sigma = \sigma^2\varrho. \tag{110}$$

Auch muß

$$p\varrho\sigma = p\sigma\varrho \tag{111}$$

gelten, denn sonst wäre $[p\varrho, \sigma] = p[\varrho, \sigma] \neq 0$, also $R = \{p\varrho, \sigma\}$ im Gegensatz zur Minimaleigenschaft von m.

Da ϱ, σ nilpotent sind, so haben ($109_{3,4}$) die Gleichungen $\varrho^r = 0$, $\sigma^s = 0$ zur Folge. Hiernach besagen (109), (110), (111), daß die Gleichungen (2), (3), (4) erfüllt sind, daß also R wegen (108) ein homomorphes Bild von R_2 ist. Das beweist Satz 1′.

§ 8. Beweis der Sätze 2′, 3′

Nach den schon bewiesenen Sätzen 2, 3 sind alle Ringe R_2, R_3 nichtnilpotent und einstufig nichtkommutativ, ferner unterscheiden sich diese voneinander darin, daß R_2 keine einseitigen Nullteiler, bzw. R_3 auch einseitige Nullteiler enthält. Letztere Eigenschaft hat auch der zu R_3 antiisomorphe Ring, der freilich auch einstufig nichtkommutativ ist. Wir legen uns jetzt umgekehrt einen beliebigen nichtnilpotenten einstufig nichtkommutativen endlichen Ring R vor und beweisen, daß R entweder ein R_2 oder ein R_3 oder antiisomorph zu einem R_3 ist. Nach dem gesagten werden wir hierdurch beide Sätze 2′, 3′ bewiesen haben.

Wir wissen, daß R ein p-Ring sein muß.

Mit $\mathfrak{n}$ bezeichnen wir das Radikal von R. Da R nicht nilpotent ist, so ist $\mathfrak{n} \neq R$.

Bekanntlich ist $R/\mathfrak{n}$ radikalfrei, also wegen der Endlichkeit von R sogar halbeinfach. Nach dem Struktursatz von Wedderburn—Artin gilt also eine direkte Summenzerlegung

$$R/\mathfrak{n} = K_1 \oplus \cdots \oplus K_k, \tag{112}$$

wobei die K_i volle Matrizenringe über (endlichen) Körpern sind. Da die K_i homomorphe Bilder von R sind, so sind sie entweder kommutativ oder ein-

stufig nichtkommutativ. Da aber ein voller Matrizenring vom Rang > 1 über einem Körper stets auch nichtkommutative echte Unterringe enthält, so müssen $\mathsf{K}_1, \ldots, \mathsf{K}_k$ (vom Rang 1, d. h.) lauter Körper sein.

Wir zeigen $k = 1$. Denn nehmen wir $k \geq 2$ an und bezeichnen mit $\mathsf{S}_1, \ldots, \mathsf{S}_k$ diejenigen Unterringe von R, für die

$$\mathsf{S}_i/\mathfrak{n} = \mathsf{K}_i \qquad (i = 1, \ldots, k) \tag{113}$$

gilt. Für jedes i nehmen wir ein Element σ_i aus S_i derart, daß die Restklasse

$$\varkappa_i = \sigma_i + \mathfrak{n} \tag{114}$$

ein Erzeugendes des Körpers K_i ist. Es sei $f_i(x)$ das Minimalpolynom von $\varkappa_i$, d. h. das irreduzible Hauptpolynom über dem Primkörper von K_i mit der Eigenschaft $f_i(\varkappa_i) = 0$. (Wir nehmen dabei die Koeffizienten von $f_i(x)$ als ganze Zahlen an, die nur mod p in Betracht kommen, aber wir wählen sie irgendwie fest.) Mit c_i bezeichnen wir das konstante Glied von $f_i(x)$. Bei passender Wahl von σ_i wird $\varkappa_i \neq 0$, d. h. $p \not\mid c_i$. Wir setzen

$$g_i(x) = f_i(x)\, x = h_i(x) + c_i x \qquad (i = 1, \ldots, k), \tag{115}$$

wobei also die $h_i(x)$ durch x^2 teilbare Polynome aus $\mathfrak{J}[x]$ sind. Wegen $g_i(\varkappa_i) = f_i(\varkappa_i)\,\varkappa_i = 0$ gilt

$$g_i(\sigma_i) \in \mathfrak{n} \qquad (i = 1, \ldots, k). \tag{116}$$

Da die $\mathsf{S}_1, \ldots, \mathsf{S}_k$ wegen $k \geq 2$ echte Unterringe von R sind, so sind sie kommutativ. Da sie wegen (113) $\mathfrak{n}$ enthalten, so folgt aus (116)

$$[\sigma_j, g_i(\sigma_i)] = 0 \qquad (i, j = 1, \ldots, k). \tag{117}$$

Ferner folgt aus (112) $\varkappa_i \varkappa_j = 0$ $(i \neq j)$, d. h. wegen (114) $\sigma_i \sigma_j \in \mathfrak{n}$. Da $\mathfrak{n}$ ein Ideal von R ist, so folgt hieraus

$$\sigma_i^a \sigma_j^b \in \mathfrak{n} \qquad (i \neq j;\ i, j = 1, \ldots, k;\ ab \geqq 1).$$

Hieraus, aus $\mathfrak{n} \subset \mathsf{S}_i$ und der Kommutativität von S_i folgt

$$\sigma_i^c \sigma_j = \sigma_i^{c-1} \sigma_i \sigma_j = \sigma_i \sigma_j \sigma_i^{c-1} = \sigma_j \sigma_i^{c-1} \sigma_i = \sigma_j \sigma_i^c \qquad (c \geqq 2;\ i, j = 1, \ldots, k).$$

Dies, (115), (117) und $x^2 | h_i(x)$ ergeben:

$$0 = [\sigma_j, g_i(\sigma_i)] = [\sigma_j, h_i(\sigma_i) + c_i \sigma_i] = [\sigma_j, c_i \sigma_i] = c_i [\sigma_j, \sigma_i],$$

also $[\sigma_j, \sigma_i] = 0$ $(i, j = 1, \ldots, k)$. Da andererseits nach (112), (113), (114)

$$\mathsf{R} = \{\sigma_1, \ldots, \sigma_k, \mathfrak{n}\}, \quad \mathsf{S}_i = \{\sigma_i, \mathfrak{n}\} \qquad (i = 1, \ldots, k)$$

gelten und $\mathsf{S}_1, \ldots, \mathsf{S}_k$ kommutativ sind, so folgt letzteres auch für R. Dieser Widerspruch beweist die Behauptung $k = 1$.

Hiernach und nach (112) ist $\mathsf{R}/\mathfrak{n}$ ein (endlicher) Körper, den wir fortan mit K bezeichnen:

$$\mathsf{R}/\mathfrak{n} = \mathsf{K}. \tag{118}$$

Mit ϱ bezeichnen wir ein Element von R, wofür die Restklasse $\varrho + \mathfrak{n}$ ein Erzeugendes der Gruppe K^* ist:

(119) $$\mathsf{K}^* = \{\varrho + \mathfrak{n}\}^*, \quad \text{also} \quad \mathsf{K} = \{\varrho + \mathfrak{n}\}.$$

Wir zeigen, daß der Grad von K eine Primzahlpotenz ($\neq 1$) ist.

Im anderen Fall enthielte nämlich K zwei echte Unterkörper $\mathsf{K}_1, \mathsf{K}_2$ mit

$$\mathsf{K} = \{\mathsf{K}_1, \mathsf{K}_2\}.$$

Für die Unterringe $\mathsf{S}_1, \mathsf{S}_2$ von R definiert durch

$$\mathsf{S}_1/\mathfrak{n} \cong \mathsf{K}_1, \quad \mathsf{S}_2/\mathfrak{n} \cong \mathsf{K}_2$$

gilt dann

$$\mathsf{R} = \{\mathsf{S}_1, \mathsf{S}_2\}.$$

(Diese $\mathsf{K}_1, \mathsf{K}_2, \mathsf{S}_1, \mathsf{S}_2$ haben mit den obigen bei (112), (113) nichts zu tun!) Zwei beliebige Elemente σ_1, σ_2 von S_1 bzw. S_2 lassen sich in der Form

$$\sigma_i = f_i(\varrho) + \nu_i \qquad (i = 1, 2)$$

annehmen, wobei $f_1(x), f_2(x)$ Polynome aus $\mathfrak{J}[x]_n$ sind und ν_1, ν_2 in $\mathfrak{n}$ gehören. Wegen $\mathfrak{n} \subset \mathsf{S}_1, \mathsf{S}_2 (\subset \mathsf{R})$ und der Kommutativität von $\mathsf{S}_1, \mathsf{S}_2$ gelten

$$[f_1(\varrho), \nu_2] = 0, \quad [f_2(\varrho), \nu_1] = 0.$$

Da noch mehr $[\nu_1, \nu_2] = 0$ und trivial $[f_1(\varrho), f_2(\varrho)] = 0$ gelten, so folgt

$$[\sigma_1, \sigma_2] = 0.$$

Hiernach wäre R kommutativ, ein Widerspruch, durch den wir bewiesen haben, daß K vom Primzahlpotenzgrad ist.

Dementsprechend setzen wir

(120) $$O(\mathsf{K}) = p^{q^e}$$

mit einer Primzahl q und einer nichtnegativen ganzen Zahl e. Aus (119), (120) folgt

(121) $$\varrho^{p^{q^e}} \equiv \varrho \pmod{\mathfrak{n}}.$$

Wegen (118), (119) ist $\mathsf{R} = \{\varrho, \mathfrak{n}\}$. Also gibt es ein Element σ von $\mathfrak{n}$ mit

(122) $$[\varrho, \sigma] \neq 0,$$

woraus

(123) $$\mathsf{R} = \{\varrho, \sigma\}$$

folgt.

Wir definieren rekursiv

(124) $$\sigma_1 = [\varrho, \sigma], \quad \sigma_{i+1} = [\varrho, \sigma_i] \qquad (i = 1, 2, \ldots).$$

Da $\mathfrak{n}$ ein Ideal von R ist, so gelten mit $\sigma \in \mathfrak{n}$ zusammen alle

(125) $$\sigma, \sigma_1, \sigma_2, \ldots \in \mathfrak{n}.$$

Wir beweisen

(126) $$\sigma_1, \sigma_2, \ldots \neq 0.$$

Nach (122), (124$_1$) ist nämlich $\sigma_1 \neq 0$. Um zunächst $\sigma_2 \neq 0$ zu beweisen, nehmen wir $\sigma_2 = 0$ an. Dies bedeutet nach (124)

$$\varrho(\varrho\sigma - \sigma\varrho) - (\varrho\sigma - \sigma\varrho)\varrho = 0,$$

d. h.

$$\sigma\varrho^2 = 2\varrho\sigma\varrho - \varrho^2\sigma.$$

Hieraus folgt allgemein

(127) $$\sigma\varrho^i = i\varrho^{i-1}\sigma\varrho - (i-1)\varrho^i\sigma \qquad (i = 1, 2, \ldots),$$

wie man das aus dem Induktionsschluß

$$\sigma\varrho^{i+1} = \sigma\varrho^i\varrho = i\varrho^{i-1}\sigma\varrho^2 - (i-1)\varrho^i\sigma\varrho = 2i\varrho^i\sigma\varrho - i\varrho^{i+1}\sigma - (i-1)\varrho^i\sigma\varrho =$$
$$= (i+1)\varrho^i\sigma\varrho - i\varrho^{i+1}\sigma$$

sieht. Andererseits bezeichne man den Exponenten in (121) mit p_1; dann gilt $\varrho^{p_1} \equiv \varrho \pmod{\mathfrak{n}}$, also allgemeiner

$$\varrho^{p_1^j} \equiv \varrho \pmod{\mathfrak{n}} \qquad (j = 1, 2, \ldots).$$

Wegen der Kommutativität von $\mathfrak{n}$ und wegen (122) ist also

(128) $$[\varrho^{p_1^j}, \sigma] \neq 0 \qquad (j = 1, 2, \ldots).$$

Wendet man aber (127) für ein $i = p_1^j$ mit $o^+(\sigma) \mid p_1^j$ an, so hat man

$$\sigma\varrho^{p_1^j} = \varrho^{p_1^j}\sigma.$$

Wir sind mit (128) in einen Widerspruch geraten und haben hierdurch $\sigma_2 \neq 0$ bewiesen. Dies drücken wir auch in der Form $[\varrho, [\varrho, \sigma]] \neq 0$ aus.

Um den Beweis von (126) zu beenden, nehmen wir an, daß $\sigma_i \neq 0$ für ein $i (\geq 2)$ schon bewiesen ist. Wegen $[\varrho, \sigma_{i-1}] = \sigma_i \neq 0$ und wegen (125) ist σ_{i-1} ein mit σ gleichberechtigtes Element, somit läßt sich das vorige auf σ_{i-1} statt σ anwenden. Es entsteht

$$[\varrho, [\varrho, \sigma_{i-1}]] \neq 0,$$

d. h. $\sigma_{i+1} \neq 0$. Somit ist (126) allgemein bewiesen.

Von jetzt an spezialisieren wir σ so, daß vor allem der Nilpotenzgrad $o(\sigma) (\geq 2)$, außerdem auch noch die additive Ordnung $o^+(\sigma) (= p)$ möglichst klein ist, und zeigen, daß dann

(129) $$\sigma^2 = 0, \quad p\sigma = 0$$

gelten.

Nach (125), (126) gelten nämlich $\sigma_1 \in \mathfrak{n}$ und $[\varrho, \sigma_1] \neq 0$. Folglich genügt es wegen der eben gemachten Minimalannahmen, wenn wir (129) für σ_1 statt

σ zeigen. Da nun $\mathfrak{n}$ kommutativ ist und die Elemente $\sigma, \varrho\sigma, \sigma\varrho, p\varrho$ enthält, so gelten:

$$\varrho\sigma\sigma_1 = \varrho(\sigma\varrho\sigma - \sigma^2\varrho) = \varrho(\sigma\sigma\varrho - \sigma^2\varrho) = 0,$$
$$\sigma\varrho\sigma_1 = \sigma\varrho\varrho\sigma - \sigma\varrho\sigma\varrho = \varrho\sigma\sigma\varrho - \varrho\sigma\sigma\varrho = 0,$$
$$\sigma_1^2 = \varrho\sigma\sigma_1 - \sigma\varrho\sigma_1 = 0,$$
$$p\sigma_1 = p[\varrho, \sigma] = [p\varrho, \sigma] = 0.$$

Mit den letzten zwei Gleichungen ist die Behauptung (129) bewiesen.

Wir zeigen, daß für das Hauptideal (σ) $(\subset \mathfrak{n})$ von R

(130) $$(\sigma)^2 = 0, \quad p(\sigma) = 0$$

gelten.

Aus der Kommutativität von (σ) folgt nämlich $(\sigma)^2 = (\sigma^2)$. Hiernach ist (130_1) wegen (129_1) richtig. Aus (129_2) folgt (130_2) trivial.

Wir bemerken folgendes. Obiges Element σ werden wir mehrmals durch ein anderes, mit ϱ nichtvertauschbares Element von (σ) ersetzen müssen. Das ist erlaubt, da dabei (129) wegen (130) erhalten bleibt.

Hier machen wir eine Fallunterscheidung, indem wir jetzt den (leichteren) Fall voranschicken, daß es unter allen möglichen Paaren ϱ, σ auch eins mit $\varrho\sigma = 0$ oder $\sigma\varrho = 0$ gibt. Da in diesem Fall wegen (108_2) entweder $\varrho\sigma = 0$, $\sigma\varrho \neq 0$ oder $\sigma\varrho = 0$, $\varrho\sigma \neq 0$ gilt und diese zwei Möglichkeiten ineinander übergehen, wenn man R durch den zu ihm antiisomorphen Ring ersetzt, so darf jetzt angenommen werden, daß eben

(131) $$\varrho\sigma \neq 0, \quad \sigma\varrho = 0$$

gilt. Wir werden zeigen, daß dann R ein R_3 ist.

Nach (121) gilt[6]

(132) $$\varrho - \varrho^k \in \mathfrak{n}$$

mit einem $k(\geqq 2)$, auf dessen genauen Wert es jetzt nicht ankommen wird. Da $\mathfrak{n}$ nilpotent ist, so folgt hieraus

$$(\varrho - \varrho^k)^l = 0$$

mit einem $l(\geqq 1)$. Jedenfalls gilt also eine Gleichung von der Form

$$\varrho^c = \varrho^c f(\varrho)$$

mit einer natürlichen Zahl c und einem Polynom $f(x)$ aus $\mathfrak{J}[x]_0$. Hieraus folgt sofort

$$\varrho^c = \varrho^c f(\varrho)^i \qquad (i = 1, 2, \ldots).$$

Wendet man dies mit $i = c + 1$ an, so folgt

(133) $$\varrho^c = \varrho^{2c} g(\varrho)$$

6 Den oben folgenden Schluß verdanke ich zum Teil Herrn G. Pollák.

mit einem Polynom $g(x)$ aus $\mathfrak{J}[x]_0$. Da hiernach

$$\varrho^c \varrho^i = \varrho^{2c} g(\varrho) \varrho^i \qquad (i = 0, 1, \ldots)$$

ist, so ergibt sich leicht

$$\varrho^c g(\varrho) = \varrho^{2c} g(\varrho)^2.$$

Setzt man also

(134) $$\alpha = \varrho^c g(\varrho),$$

so gilt

(135) $$\alpha^2 = \alpha.$$

Wegen (131_2), (134) gilt auch

(136) $$\sigma\alpha = 0.$$

Wir zeigen

(137) $$\alpha\sigma \neq 0.$$

Wäre nämlich $\alpha\sigma = 0$, so folgte aus (133), (134)

$$\varrho^c \sigma = \varrho^{2c} g(\varrho) \sigma = \varrho^c \alpha\sigma = 0.$$

Es sei d die kleinste natürliche Zahl mit

(138) $$\varrho^d \sigma = 0.$$

Wegen (131_1) ist gewiß $d \geqq 2$. Andererseits folgt aus (131_2)

$$\sigma(\varrho - \varrho^k) = 0.$$

Nach (125), (132) gehören beide Faktoren in $\mathfrak{u}$. Da ferner $\mathfrak{u}$ kommutativ ist, so folgt

$$(\varrho - \varrho^k)\, \sigma = 0.$$

Dieses ergibt

$$(\varrho^{d-1} - \varrho^{d+k-2})\, \sigma = 0.$$

Wegen (138) und $k \geqq 2$ gilt also $\varrho^{d-1}\sigma = 0$. Da dies der Minimaleigenschaft von d widerspricht, so ist hiermit (137) bewiesen.

Wir setzen noch

$$\beta = \alpha\sigma.$$

Nach (129_2), (131_2), (135), (136), (137) gelten dann

(139) $$p\beta = 0, \quad \alpha^2 = \alpha, \quad \beta^2 = 0, \quad \alpha\beta = \beta\,(\neq 0), \quad \beta\alpha = 0.$$

Da hiernach insbesondere $\alpha\beta \neq \beta\alpha$ ist, so gilt auch

(140) $$\mathsf{R} = \{\alpha, \beta\}.$$

Endlich setzen wir

(141) $$o^+(\alpha) = p^m$$

mit einer natürlichen Zahl m. Aus (139), (140), (141) folgt, daß

$$a\alpha + b\beta \quad (a = 0, \ldots, p^m - 1\,;\ b = 0, \ldots, p-1)$$

die sämtlichen verschiedenen Elemente von R sind, also

(142) $$O(\mathsf{R}) = p^{m+1}$$

gilt. Da andererseits nach (141)

$$p^m \alpha = 0$$

ist, so sehen wir hieraus und aus (139), (140), daß R ein homomorphes Bild des (durch (8) und (9) definierten) Ringes R_3 ist:

$$R_3 \sim R.$$

Da aber nach (25), (142) $O(R_3) = O(R)$ ist, so folgt sogar, daß R (bis auf Isomorphie) mit R_3 übereinstimmt, wie wir es behauptet haben.

Es ist noch der andere Fall übrig, daß nämlich für alle möglichen Paare ϱ, σ beide Ungleichungen

$$\varrho\sigma \neq 0, \quad \sigma\varrho \neq 0 \tag{143}$$

gelten. Es wird sich zeigen, daß dann unser Ring R ein R_2 ist.

Vor allem zeigen wir, daß jetzt sogar

$$\varrho^i \sigma \varrho^j \neq 0 \qquad (i, j = 0, 1, \ldots) \tag{144}$$

gilt.

Dann nehmen wir an, daß (144) für ein Paar i, j falsch ist:

$$\varrho^i \sigma \varrho^j = 0,$$

wobei freilich $i + j > 0$ sein muß. Ist dabei $i > 0$, so muß auch

$$\varrho^{i-1} \sigma \varrho^{j+1} = 0$$

sein, da sonst $\bar{\sigma} = \varrho^{i-1} \sigma \varrho^j$ ein Element von $\mathfrak{n}$ mit $\varrho\bar{\sigma} = 0$, $\bar{\sigma}\varrho \neq 0$ wäre, was wegen der Annahme bei (143) unmöglich ist. Ähnlich zeigt man, daß im Fall $j > 0$ auch

$$\varrho^{i+1} \sigma \varrho^{j-1} = 0$$

gelten muß. Mit Induktion folgt, daß alle

$$\varrho^a \sigma \varrho^{i+j-a} = 0 \qquad (a = 0, \ldots, i + j)$$

gelten, woraus wegen der Definition (124) offenbar $\sigma_{i,j} = 0$ folgt. Da dies nach (126) unmöglich ist, so ist (144) richtig.

Da σ_1 (wegen (126)) ein mit σ gleichberechtigtes Element ist, so folgt aus (144) auch noch

$$\varrho^i \sigma_1 \varrho^j \neq 0,$$

d. h.

$$[\varrho, \varrho^i \sigma \varrho^j] \neq 0 \qquad (i, j = 0, 1, \ldots). \tag{145}$$

Mit $F(x)$ bezeichnen wir das Minimalpolynom des Elementes $\varrho + \mathfrak{n}$ von K, wobei also $F(x)$ ein mod p irreduzibles Hauptpolynom aus $\mathfrak{J}[x]$ vom Grad q^e ist, wofür (vgl. (119), (120))

$$F(\varrho + \mathfrak{n}) = 0 \tag{146}$$

gilt. Wir setzen

$$G(x) = F(x)\, x. \tag{147}$$

Aus (146) folgt dann offenbar

$$G(\varrho) \in \mathfrak{n}. \tag{148}$$

Wegen der Nilpotenz von $\mathfrak{n}$ gibt es also eine natürliche Zahl r mit

$$G(\varrho)^r = 0. \tag{149}$$

Wir beweisen, daß bei passender Wahl von σ

$$G(\varrho)\, \sigma = 0 \tag{150}$$

ist.

Wegen (149) gibt es nämlich ein $i (\geqq 1)$ mit

$$G(\varrho)^i \sigma = 0, \quad G(\varrho)^{i-1} \sigma \neq 0.$$

Im Fall $i = 1$ folgt hieraus (150). Wir betrachten den Fall $i \geqq 2$ und setzen unsere Behauptung für die kleineren i voraus. Ist

$$[\varrho, G(\varrho)^{i-1} \sigma] \neq 0,$$

so ist $\bar{\sigma} = G(\varrho)^{i-1} \sigma$ ein mit σ gleichberechtigtes Element; da dabei $G(\varrho)\bar{\sigma} = 0$ ist, so ist jetzt die Behauptung über (150) richtig. Im anderen Fall ist (nach (124_1))

$$0 = [\varrho, G(\varrho)^{i-1} \sigma] = G(\varrho)^{i-1} [\varrho, \sigma] = G(\varrho)^{i-1} \sigma_1 .$$

Nach (125), (126) ist σ_1 ein mit σ gleichberechtigtes Element. Ersetzen wir also σ durch σ_1, so tritt an Stelle von i eine kleinere Zahl ein, womit wegen der Annahme die Behauptung über (150) bewiesen ist.

Aus (148), (150) und der Kommutativität von $\mathfrak{n}$ folgt

$$\sigma G(\varrho) = 0. \tag{151}$$

Ferner ergibt sich aus (147), (150), (151):

$$F(\varrho) \varrho \sigma \varrho = 0, \quad \varrho \sigma \varrho F(\varrho) = 0.$$

Ersetzt man also σ durch $\varrho\sigma\varrho$, was wegen (des Falls $i = j = 1$ von) (145) erlaubt ist, so gelten sogar

$$F(\varrho)\, \sigma = 0, \quad \sigma F(\varrho) = 0, \tag{152}$$

was wir fortan ebenfalls annehmen wollen.

Als eine Verschärfung von (152) beweisen wir für Elemente $\bar{\sigma} (\neq 0)$ aus dem Ideal (σ) und Polynome $f(x)$ aus $\mathfrak{J}[x]$ die Regel:

$$f(\varrho)\bar{\sigma} = 0 \iff \bar{\sigma} f(\varrho) = 0 \iff f(x) \equiv 0 \pmod{p, F(x)}. \tag{153}$$

Es genügt hiervon

$$f(\varrho)\, \bar{\sigma} = 0 \iff f(x) \equiv 0 \pmod{p, F(x)}$$

zu beweisen. Besteht die rechte Seite, so besteht wegen (130_2) und (152_1) die

linke Seite. Besteht die rechte Seite nicht, so gibt es — da $F(x) \bmod p$ irreduzibel ist — drei Polynome $a(x)$, $b(x)$, $c(x)$ aus $\mathfrak{J}[x]$ mit

$$f(x)\,a(x)+p\,b(x)+F(x)\,c(x)=1.$$

Da nach (130_2) und (152_2) $p\sigma=0$, $F(\varrho)\sigma=0$ gelten, so folgt hieraus $f(\varrho)\,a(\varrho)\bar{\sigma}=\bar{\sigma}$, $f(\varrho)\bar{\sigma}\neq 0$. Somit ist (153) bewiesen.

Wir zeigen

$$e>0 \tag{154}$$

(was nach (120) und (146) bedeutet, daß der gemeinsame Grad q' von K und $F(x)$ mindestens q ist).

Im Fall $e=0$ ist nämlich nach (120) $O(\mathrm{K})=p$, also nach (146) $F(x)=x+c$ mit einer ganzen Zahl c (die übrigens zu p prim ist). Hieraus und aus (152) folgt

$$(\varrho+c)\,\sigma=0, \quad \sigma\,(\varrho+c)=0,$$

also $\varrho\sigma=\sigma\varrho$. Dieser Widerspruch beweist (154).

Wir setzen zur Abkürzung

$$Q=q^e. \tag{155}$$

Nach der Definition von $F(x)$ (bei (146)) ist dann

$$F(x)=x^Q+a_1x^{Q-1}+\cdots+a_Q \tag{156}$$

mit Koeffizienten aus $\mathfrak{J}$.

Wir nehmen die zu $F(x)$ gehörenden Hornerschen Polynome

$$F_i(x)=x^i+a_1x^{i-1}+\cdots+a_i \quad (i=0,\ldots,Q;\ F_0(x)=1,\ F_Q(x)=F(x))$$

zu Hilfe und setzen

$$\tau_j=\sigma\varrho^{Q-1}+F_1(\varrho^{p^j})\,\sigma\varrho^{Q-2}+\cdots+F_{Q-1}(\varrho^{p^j})\,\sigma \qquad (j=1,\ldots,Q-1). \tag{157}$$

Die $\tau_1,\ldots,\tau_{Q-1}$ sind lauter Elemente des Ideals (σ).

Wegen $xF_i(x)=F_{i+1}(x)-a_{i+1}$ $(i=0,\ldots,Q-1)$ folgt aus (157)

$$\varrho^{p^j}\,\tau_j=(F_1(\varrho^{p^j})-a_1)\,\sigma\varrho^{Q-1}+\cdots+(F_Q(\varrho^{p^j})-a_Q)\,\sigma,$$

d. h.

$$\varrho^{p^j}\,\tau_j=\tau_j\varrho-\sigma F(\varrho)+F(\varrho^{p^j})\,\sigma \qquad (j=1,\ldots,Q-1). \tag{158}$$

Da $F(x^{p^j})\equiv F(x)^{p^j} \pmod p$ ist, so folgt aus (129_2) und (152) das Verschwinden der letzten zwei Glieder in (158). Folglich ist

$$\tau_j\varrho=\varrho^{p^j}\,\tau_j \qquad (j=1,\ldots,Q-1). \tag{159}$$

Wir beweisen, daß nicht alle $\tau_1,\ldots,\tau_{Q-1}$ verschwinden.

Denn nehmen wir $\tau_1=\cdots=\tau_{Q-1}=0$ an und setzen zur Abkürzung

$$\varrho_i=\varrho^{p^i} \qquad (i=1,\ldots,Q-1), \tag{160}$$

$$\xi_k=\sigma\varrho^{Q-k-1} \qquad (k=0,\ldots,Q-1). \tag{161}$$

Wegen (157) lautet dann die Annahme so:

(162) $$\xi_0 + F_1(\varrho_i)\xi_1 + \cdots + F_{Q-1}(\varrho_i)\xi_{Q-1} = 0 \qquad (i = 1, \ldots, Q-1).$$

Die Koeffizientenmatrix dieses homogen linearen Gleichungssystems ist

$$\begin{pmatrix} 1 & F_1(\varrho_1) & \cdots F_{Q-1}(\varrho_1) \\ \vdots & & \\ 1 & F_1(\varrho_{Q-1}) & \cdots F_{Q-1}(\varrho_{Q-1}) \end{pmatrix}.$$

Nach Streichen der k-ten Spalte entsteht eine quadratische Matrix; ihre Determinante bezeichnen wir mit D_k $(k = 1, \ldots, Q)$. Aus (162) folgt dann

(163) $$D_Q\xi_{Q-2} + D_{Q-1}\xi_{Q-1} = 0.$$

Werden in D_Q passende Vielfache von Spalten aus späteren Spalten subtrahiert, so gewinnen wir eine Vandermondesche Determinante bestehend aus den Zeilen

$$1, \varrho_i, \ldots, \varrho_i^{Q-2} \qquad (i = 1, \ldots, Q-1),$$

weshalb

(164) $$D_Q = \prod_{1 \leq i < j \leq Q-1} (\varrho_j - \varrho_i)$$

ist. Nach ähnlicher Behandlung von D_{Q-1} gewinnen wir eine Determinante bestehend aus den Zeilen

$$1, \varrho_i, \ldots, \varrho_i^{Q-3}, \ \varrho_i^{Q-1} + a_1\varrho_i^{Q-2} \qquad (i = 1, \ldots, Q-1).$$

Leicht bekommen wir hieraus

$$D_{Q-1} = D_Q(\varrho_1 + \cdots + \varrho_{Q-1} + a_1).$$

Wegen (161), (163) gilt also

(165) $$D_Q(\sigma\varrho + (\varrho_1 + \cdots + \varrho_{Q-1} + a_1)\sigma) = 0.$$

Nun ist nach (160), (164)

$$D_Q = \Delta(\varrho) \quad \text{mit} \quad \Delta(x) = \prod_{1 \leq i < j \leq Q-1} (x^{p^j} - x^{p^i}).$$

Da $F(x) \bmod p$ irreduzibel und vom Grad Q ist, so ist bekanntlich

$$\Delta(x) \not\equiv 0 \pmod{p, F(x)}.$$

Hieraus und aus (165) folgt nach der Regel (153)

(166) $$\sigma\varrho + (\varrho_1 + \cdots + \varrho_{Q-1} + a_1)\sigma = 0.$$

Wieder wegen der gesagten Eigenschaft von $F(x)$ und wegen (156) gilt für das Polynom

(167) $$s(x) = x + x^p + x^{p^2} + \cdots + x^{p^{Q-1}} + a_1$$

die Kongruenz

$$s(x) \equiv 0 \pmod{p, F(x)}.$$

Wegen (129_2) und (152_1) folgt hieraus

$$s(\varrho)\,\sigma = 0.$$

Andererseits ist nach (160), (167)

$$s(\varrho) = \varrho + \varrho_1 + \cdots + \varrho_{Q-1} + a_1,$$

also

$$(\varrho_1 + \cdots + \varrho_{Q-1} + a_1)\,\sigma = -\varrho\sigma.$$

Hieraus und aus (166) folgt $\sigma\varrho - \varrho\sigma = 0$. Dieser Widerspruch beweist die Behauptung, daß nicht alle $\tau_1, \ldots, \tau_{Q-1}$ verschwinden.

Wir nehmen also ein beliebiges nichtverschwindendes τ_t $(0 < t < Q)$ und bezeichnen es mit τ. Nach (157), (159) gelten dann

(168) $$\tau\varrho = \varrho^{p^t}\tau, \quad \tau \in (\sigma), \quad 0 < t < Q.$$

Für dieses τ zeigen wir

(169) $$[\varrho, \tau] \neq 0.$$

Da nämlich wegen (155), (168_3)

$$x^{p^t} - x \not\equiv 0 \pmod{p, F(x)}$$

ist, so folgt aus (153), (168_2)

$$(\varrho^{p^t} - \varrho)\,\tau \neq 0.$$

Dies besagt wegen (168_1) die Richtigkeit von (169).

Wegen (168_2) und (169) ist τ ein mit σ gleichberechtigtes Element. Deshalb dürfen wir das bisherige σ durch dieses τ ersetzen, das wir dann wieder mit σ bezeichnen. Nach (168) haben wir dann

(170) $$\sigma\varrho = \varrho^{p^t}\sigma, \quad 0 < t < Q.$$

(Freilich behalten (129), (152) und sogar (153) für dieses neue σ ihre Gültigkeit.)

Wir beweisen

(171) $$q^{e-1} \mid t.$$

Als Vorbereitung zeigen wir die Regel

(172) $$\{\varrho^s, \mathfrak{n}\} = \mathfrak{R} \iff \{\varrho^s, \sigma\} = \mathfrak{R}$$

gültig für jede natürliche Zahl s. Aus der rechten Seite folgt nämlich die linke Seite trivial. Wir nehmen dann an, daß die rechte Seite falsch ist. Dann sind ϱ^s, σ miteinander vertauschbar. Wegen (123) folgt hieraus, daß ϱ^s im Zentrum von $\mathfrak{R}$ liegt. Da ferner $\mathfrak{n}$ ein kommutativer Unterring von $\mathfrak{R}$ ist, so ist auch $\{\varrho^s, \mathfrak{n}\}$ kommutativ, d. h. ungleich $\mathfrak{R}$. Somit haben wir (172) bewiesen.

Wir wollen die Regel (172) umformen. Ihre linke Seite ist wegen (118) gleichbedeutend mit $\{\varrho^s, \mathfrak{n}\}/\mathfrak{n} = \mathfrak{K}$. Dies ist wieder gleichbedeutend damit, daß

die Restklasse

$$\varrho^s + \mathfrak{n} = (\varrho + \mathfrak{n})^s$$

ein Erzeugendes von K ist. Dies ist wegen (119), (120) bekanntlich gleichbedeutend mit

$$\frac{p^{q'}-1}{p^{q^{e-1}}-1} \nmid s.$$

Andererseits gilt für die rechte Seite von (172) wegen (153), (170):

$$\{\varrho^s, \sigma\} = \mathsf{R} \Longleftrightarrow [\varrho^s, \sigma] \neq 0 \Longleftrightarrow (\varrho^s - \varrho^{sp^t})\,\sigma \neq 0 \Longleftrightarrow$$

$$\Longleftrightarrow x^s - x^{sp^t} \not\equiv 0 \pmod{p, F(x)} \Longleftrightarrow x^{s(p^t-1)} - 1 \not\equiv 0 \pmod{p, F(x)} \Longleftrightarrow$$

$$\Longleftrightarrow p^{q'} - 1 \nmid s\,(p^t - 1) \Longleftrightarrow \frac{p^{q'}-1}{p^{(q',t)}-1} \nmid s,$$

wobei berücksichtigt wurde, daß $F(x) \bmod p$ irreduzibel und vom Grad q', daß ferner stets $(p^a-1, p^b-1) = p^{(a,b)}-1$ ist (a, b natürliche Zahlen). Hiernach ist (172) gleichbedeutend mit $q^{e-1} = (q', t)$, woraus (171) folgt.

Um das spätere nicht unterbrechen zu brauchen, beweisen wir hier den folgenden

Hilfssatz 3. *Die sämtlichen Ideale von $\mathfrak{J}[x]_0$ sind die*

$$\mathfrak{a}_0 = x\mathfrak{a},$$

wobei $\mathfrak{a}$ alle Ideale von $\mathfrak{J}[x]$ durchläuft.

Es ist nämlich klar, daß jedes $x\mathfrak{a}$ ein Ideal von $\mathfrak{J}[x]_0$ ist. Umgekehrt, betrachten wir ein Ideal $\mathfrak{a}_0$ von $\mathfrak{J}[x]_0$. Dieses läßt sich in der Form $x\mathfrak{a}$ schreiben, wobei jetzt $\mathfrak{a}$ eine Teilmenge von $\mathfrak{J}[x]$ bezeichnet. Wir haben zu zeigen, daß $\mathfrak{a}$ sogar ein Ideal von $\mathfrak{J}[x]$ ist. Aus der Moduleigenschaft von $\mathfrak{a}_0$ folgt sofort die von $\mathfrak{a}$. Wir betrachten zwei Polynome

$$f(x)\,(\in \mathfrak{a}), \quad g(x)\,(\in \mathfrak{J}[x]).$$

Wenn uns gelingt $f(x)\,g(x) \in \mathfrak{a}$ auszuweisen, so werden wir Hilfssatz 3 bewiesen haben. Das ist leicht. Denn setzen wir

$$g(x) = c + x\,h(x)$$

mit einem $c\,(\in \mathfrak{J})$ und $h(x)\,(\in \mathfrak{J}[x])$ an. Es gelten

$$f(x)\,g(x) = c f(x) + x f(x)\,h(x),$$

$c f(x) \in \mathfrak{a}$, $x f(x) \in \mathfrak{a}_0$. Wegen letzteres ist $x f(x)\,h(x) \in \mathfrak{a}_0$. Noch mehr gilt dann $x f(x)\,h(x) \in \mathfrak{a}$, woraus die Behauptung folgt.

Fortan bezeichnen wir mit $\mathfrak{a}_0$ das Ideal von $\mathfrak{J}[x]_0$ bestehend aus denjenigen Polynomen $f(x)\,(\in \mathfrak{J}[x]_0)$, für die $f(\varrho) = 0$ gilt. Wir wollen $\mathfrak{a}_0$ erforschen.

Bevor wir das tun, machen wir einige Bemerkungen. Wegen (123), (129_1), (170_1) lassen sich alle Elemente von $\mathfrak{R}$ in der Form

(173) $$a(\varrho)+b(\varrho)\sigma \qquad (a(x)\in\mathfrak{J}[x]_0,\ b(x)\in\mathfrak{J}[x])$$

angeben. Hierfür beweisen wir die Regel

(174) $$a(\varrho)+b(\varrho)\sigma=0 \iff a(x)\equiv 0 \ (\mathrm{mod}\ \mathfrak{a}_0),\quad b(x)\equiv 0\ (\mathrm{mod}\ p, F(x)).$$

Wegen der Definition von $\mathfrak{a}_0$ und der Regel (153) genügt es zu zeigen, daß aus der linken Seite von (174) das Verschwinden beider Glieder $a(\varrho)$, $b(\varrho)\sigma$ folgt.

Zu diesem Zweck nehmen wir

$$a(\varrho)+b(\varrho)\sigma=0,\quad b(\varrho)\sigma\neq 0$$

an. Aus dem zweiten folgt nach (153)

$$b(x)\not\equiv 0 \quad (\mathrm{mod}\ p, F(x)).$$

Also gibt es ein Polynom $c(x)$ ($\in\mathfrak{J}[x]$) mit

$$c(x)\,b(x)\equiv 1 \quad (\mathrm{mod}\ p, F(x)).$$

Dann ist wieder nach (153) $c(\varrho)\,b(\varrho)\,\sigma=\sigma$, also folgt

$$c(\varrho)\,a(\varrho)+\sigma=0.$$

Dies ist aber wegen $[\varrho,\sigma]\neq 0$ unmöglich, also ist (174) richtig.

Eine weitere Bemerkung ist die unmittelbar aus (170_1) folgende Vertauschungsregel:

$$\sigma f(\varrho)=f(\varrho^{p^t})\,\sigma \qquad (f(x)\in\mathfrak{J}[x]_0),$$

wofür man wegen (129_2) auch

(175) $$\sigma f(\varrho)=f(\varrho)^{p^t}\sigma$$

schreiben kann.

Unsere letzte vorbereitende Bemerkung ist, daß aus der Annahme

(176) $$f(x)^{p^t}\not\equiv f(x) \quad (\mathrm{mod}\ p, F(x)) \qquad (f(x)\in\mathfrak{J}[x]_0)$$

die Existenz eines Polynoms $g(x)$ mit

(177) $$g(f(x))\equiv x \quad (\mathrm{mod}\ \mathfrak{a}_0) \qquad (g(x)\in\mathfrak{J}[x]_0)$$

folgt.

Aus (176) folgt nämlich nach (174) und (175) $\sigma f(\varrho)-f(\varrho)\sigma\neq 0$, also $\mathfrak{R}=\{f(\varrho),\sigma\}$, $\varrho\in\{f(\varrho),\sigma\}$. Wegen letzteres kommt man mit ähnlicher Begründung wie bei (173) zu einer Gleichung

$$\varrho=g(f(\varrho))+b(\varrho)\sigma$$

mit zwei Polynomen $g(x)$ ($\in\mathfrak{J}[x]_0$), $b(x)$ ($\in\mathfrak{J}[x]$). Das zweite Glied der rechten Seite verschwindet wegen (174), woraus (177) folgt.

Nunmehr wollen wir das Ideal $\mathfrak{a}_0$ bestimmen. Wir setzen

(178) $$o^+(\varrho)=p^m$$

mit einer natürlichen Zahl m. Wegen $p^m \varrho = 0$ ist dann $p^m x$ in $\mathfrak{a}_0$ enthalten. Wegen (148) und der Nilpotenz von $\mathfrak{n}$ gibt es ferner eine natürliche Zahl r mit $G(\varrho)^r = 0$, d. h.

$$G(x)^r \in \mathfrak{a}_0. \tag{179}$$

Bezeichnen wir einen Augenblick mit $\mathfrak{a}$ das nach Hilfssatz 3 durch $\mathfrak{a}_0$ bestimmte Ideal von $\mathfrak{J}[x]$, so folgt $p^m, G(x)^r \in \mathfrak{a}$. Da $G(x)$ (also auch $G(x)^r$) ein Hauptpolynom ist, gilt nach einem Kronecker—Henselschen Satz (vgl. Rédei [9]), daß $\mathfrak{a}_0$ von der Form

$$\mathfrak{a}_0 = (xA_0(x),\, p^{k_1}xA_1(x),\, \ldots,\, p^{k_{s-1}}xA_{s-1}(x),\, p^{k_s}x) \tag{180}$$

ist, wobei

$$0 < k_1 < \cdots < k_s \tag{181}$$

gilt, $A_i(x)$ $(i = 0, \ldots, s-1)$ ein Hauptpolynom vom Grad n_i aus $\mathfrak{J}[x]$ ist mit

$$n_0 > n_1 > \cdots > n_{s-1} > 0 \tag{182}$$

und die Kongruenzen

$$A_i(x) \equiv 0 \pmod{p, A_{i+1}(x)} \qquad (i = 0, \ldots, s-2) \tag{183}$$

gelten.

Wegen (180) gilt

$$(\mathfrak{a}_0, px) = (xA_0(x), px). \tag{184}$$

Hieraus und aus (179) ergibt sich

$$G(x)^r \equiv 0 \pmod{xA_0(x), px}.$$

Wegen (147) gilt also

$$xA_0(x) \equiv x^a F(x)^b \pmod{px} \tag{185}$$

mit zwei natürlichen Zahlen a, b.

Wir beweisen

$$u = v = 1. \tag{186}$$

Zu diesen Zweck beachten wir, daß offenbar

$$x^{2(p^t-1)} \not\equiv 1 \pmod{p, F(x)}$$

gilt, weshalb (176) mit $f(x) = x^2$ erfüllt ist. Nach (177) gibt es also ein Polynom $g(x)$ $(\in \mathfrak{J}[x]_0)$ mit

$$g(x^2) \equiv x \pmod{\mathfrak{a}_0}.$$

Nach (184), (185) gilt dann noch mehr

$$g(x^2) \equiv x \pmod{x^a, px}.$$

Dies ist für $u \geqq 2$ unmöglich, also muß $u = 1$ gelten.

Um den Beweis von (186) zu beenden, nehmen wir $\nu \geqq 2$ an. Wir nehmen ferner ein Polynom $a(x)$ mit

(187) $$a(x)^{p^t} \mp a(x) \pmod{p, F(x)}, \quad a(x) \in \mathfrak{J}[x]_0.$$

Außerdem bestimmen wir ein Polynom $b(x)$ mit

(188) $$a'(x) + F'(x)\, b(x) \equiv 0 \pmod{p, F(x)}, \quad b(x) \in \mathfrak{J}[x]_0,$$

wobei „ʹ" den Differentialquotienten bezeichnet. Wir setzen

(189) $$f(x) = a(x) + F(x)\, b(x).$$

Hierfür ist wegen (187) die Bedingung (176) erfüllt, also gibt es ein Polynom $g(x)$ mit (177). Wegen (184), (185) und $\nu \geqq 2$ gilt dann noch mehr

$$g(f(x)) \equiv x \pmod{x\, F(x)^2, px}, \quad g(x) \in \mathfrak{J}[x]_0.$$

Hieraus folgt nach Differenzieren

(190) $$f'(x)\, g'(f(x)) \equiv 1 \pmod{F(x), p}.$$

Andererseits ist aber nach (188), (189)

$$f'(x) \equiv a'(x) + F'(x)\, b(x) \equiv 0 \pmod{p, F(x)}.$$

Wir sind mit (190) in einen Widerspruch geraten, womit $\nu = 1$, also auch (186) bewiesen ist.

Hiernach lautet (185) so:

$$x A_0(x) \equiv x F(x) \pmod{px}.$$

Da aber $F(x)$ nur $\bmod\, p$ in Betracht kommt, so läßt sich

$$A_0(x) = F(x)$$

annehmen. Da ferner $F(x)$ $\bmod\, p$ irreduzibel und $A_1(x)$ von kleinerem Grad als $A_0(x)$ ist, so folgt aus (183) notwendig $A_1(x) = 1$. Also gilt in (180) $s = 1$. Hiernach und nach (178) haben wir also

(191) $$\mathfrak{a}_0 = (x F(x), p^m x).$$

Hieraus und aus der Definition von $\mathfrak{a}_0$ folgt $F(\varrho)\varrho = 0$, $p^m \varrho = 0$. Wegen (123), (129), (152_1), (170), (171) ist also R das homomorphe Bild eines durch (6) und (7) definierten Ringes R_2:

(192) $$\mathsf{R}_2 \sim \mathsf{R}.$$

Aus (173), (174), (191) folgt sofort

(193) $$O(\mathsf{R}) = p^{(m+1)\, q^r},$$

wobei berücksichtigt wurde, daß $F(x)$ vom Grad $Q = q^r$ ist. Wegen (17), (193) gilt (192) sogar als Isomorphie. Das beendet den Beweis der Sätze 2′, 3′.

§ 9. Bemerkungen

Das Problem der Bestimmung der untereinander nichtisomorphen homomorphen Bilder einer Struktur S nennen wir das Homomorphieproblem von S. Dann besagt Satz 1′, daß die effektive Aufstellung aller verschiedenen nilpotenten einstufig nichtkommutativen endlichen Ringe auf das Homomorphieproblem der Ringe R_1 hinausläuft. Die allgemeine Lösung dieses speziellen Homomorphieproblems scheint uns eine interessante aber schwierige Aufgabe zu sein.

Z. B. betrachten wir den Ring R_1 für den einfachsten Fall $m=n=1$, $r=s=2$. Nach (1) bis (4) ist jetzt R_1 durch $R_1=\{\varrho,\sigma\}$

$$p\varrho=p\sigma=\varrho^2=\sigma^2=\varrho\sigma\varrho=\sigma\varrho\sigma=0,\quad p\sigma\varrho=p\varrho\sigma \tag{194}$$

definiert. Nach (12) ist

$$O(R_1)=p^4.$$

Nach (13) und (14) bilden $\varrho, \sigma, \varrho\sigma, \sigma\varrho$ eine Basis von R_1^+ mit

$$o^+(\varrho)=o^+(\sigma)=o^+(\varrho\sigma)=o^+(\sigma\varrho)=p.$$

Mit S bezeichnen wir ein nichtkommutatives homomorphes Bild von R_1, das nicht isomorph zu R_1 ist. Man darf annehmen, daß S aus R_1 so entsteht, daß man den Definitionsgleichungen (194) mindestens eine Gleichung von der Form

$$a\varrho+b\sigma+c\varrho\sigma+d\sigma\varrho=0\quad (a,b,c,d=0,\ldots,p-1) \tag{195}$$

hinzufügt, in der nicht $a=b=c=d=0$ ist. Wir zeigen, daß notwendig $a=b=0$ ist. Multipliziert man nämlich (195) mit σ von rechts oder von links, so entsteht nach (194) $a\varrho\sigma=0$ bzw. $a\sigma\varrho=0$. Im Fall $a\neq 0$ folgt hieraus $\varrho\sigma=0$, $\sigma\varrho=0$. Dies ist falsch, da S nicht kommutativ ist. Also ist $a=0$. Genau so bekommt man $b=0$. Im verbliebenen (195) darf aus Symmetriegründen $d\neq 0$ angenommen werden und dann hat man es mit einer Gleichung

$$\sigma\varrho=k\varrho\sigma \tag{196}$$

zu tun. Eine weitere Gleichung kommt nicht in Frage, da S nicht kommutativ ist. Um die Abhängigkeit von k zum Ausdruck zu bringen, schreiben wir S_k statt S. Dann ist S_k durch $S_k=\{\varrho,\sigma\}$ und die Gleichungen (194), (195) definiert. Es gilt $O(S_k)=p^3$. Setzt man $\tau=\varrho\sigma$, so folgt aus (196), daß ϱ, σ, τ eine Basis von S_k ist und für diese Basiselemente die Multiplikationstafel gilt:

$$\begin{array}{c|ccc} & \varrho & \sigma & \tau \\ \hline \varrho & 0 & \tau & 0 \\ \sigma & k\tau & 0 & 0 \\ \tau & 0 & 0 & 0 \end{array}$$

Es werde der leichte Beweis dem Leser überlassen, daß S_0, S_{p-1} nicht isomorph, dagegen $S_1, \ldots, S_{p-1}$ isomorph sind. Die sämtlichen, als homomorphe Bilder von R_1 entstehenden, untereinander nichtisomorphen einstufig nichtkommutativen Ringe sind somit R_1, S_0, S_{p-1}. Man sieht auch, daß S_0 und S_{p-1} die einzigen nilpotenten einstufig nichtkommutativen Ringe p^3-ter Ordnung sind.

Die Ringe R_2 verhalten sich im Fall $m = 1$ besonders einfach. Man nehme den endlichen Körper K von der Ordnung p^{q^e}, bilde die Menge aller Paare (α, β) $(\alpha, \beta \in K)$ und definiere in ihr die beiden Verknüpfungen

$$(\alpha, \beta) + (\gamma, \delta) = (\alpha + \gamma, \beta + \delta), \quad (\alpha, \beta)(\gamma, \delta) = (\alpha\gamma, \alpha\delta + \beta\gamma^P) \quad (P = p^{mq^{e-1}}).$$

So entsteht ein zu R_2 isomorpher Ring, wie man das sofort sieht. Es ist interessant, daß die endlichen Körper auch in die einstufig nichtkommutativen endlichen Gruppen stark hineinspielen (vgl. Rédei [5] oder [6]).

Eine A-Struktur nennen wir k-stufig nichtkommutativ ($k \geqq 2$), wenn alle echten A-Unterstrukturen von ihr kommutativ oder höchstens $k-1$-stufig nichtkommutativ sind und mindestens eine von diesen $k-1$-stufig nichtkommutativ ist. Die 2-stufig nichtkommutativen endlichen Gruppen lassen sich restlos bestimmen, worauf ich ein andermal zurückkommen will. Leicht läßt sich zeigen, daß alle 2-stufig nichtkommutativen endlichen Ringe nichteinfach sind; jedoch scheint ihre Bestimmung mit großen Schwierigkeiten verbunden zu sein, wohl aus dem Grunde, daß die Lösung des Homomorphieproblems der Ringe R_1 aussteht.

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SUR LA DÉCOMPOSITION DE L'ESPACE EUCLIDIEN EN ENSEMBLES HOMOGÈNES

Par

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1. Selon EMILE BOREL, un ensemble E de l'espace euclidien à n dimensions $\mathcal{E}^n$ est dit homogène si, X et Y étant deux points quelconques de E, la translation XY transforme E en lui-même.

Le continu et l'ensemble des nombres rationnels sont des exemples triviaux d'ensembles homogènes linéaires. EMILE BOREL construit [1] deux exemples d'ensembles linéaires, homogènes, non dénombrables, autres que le continu et conclut: „Les ensembles que nous avons définis sont de mesure nulle; la question reste ouverte de savoir si le continu peut être décomposé en un nombre fini ou en une infinité dénombrable d'ensembles homogènes égaux, qui ne pourraient être de mesure nulle si leur nombre est fini et ne seraient pas mesurables si leur infinité est dénombrable".

En ce qui concerne la décomposition de la droite en une infinité de puissance $\mathfrak{m}$ ($\aleph_0 \leq \mathfrak{m} \leqq 2^{\aleph_0}$) d'ensembles homogènes superposables par translation, le problème est résolu par l'affirmative (bien que non explicitement), par S. RUZIEWICZ [2] et I. HALPERIN [3].

I. HALPERIN démontre qu'il existe une décomposition de la droite en $\mathfrak{m}$ ($\aleph_0 \leq \mathfrak{m} \leqq 2^{\aleph_0}$) ensembles E_i superposables par translation et saturément non mesurables (c'est-à-dire pour tout E mesurable on a $\mu^*(E_i \cap E) = \mu(E)$, $\mu_*(E_i \cap E) = 0$, μ^*, μ et μ_* étant, respectivement, la mesure extérieure, la mesure, la mesure intérieure). On peut montrer que les ensembles E_i sont homogènes.

Les résultats contenus dans la présente Note peuvent être résumés comme il suit:

Il n'existe aucune décomposition de $\mathcal{E}^n$ en un nombre fini > 1 d'ensembles homogènes non vides. Le caractère saturé de la non mesurabilité (voir, par exemple, la décomposition de I. HALPERIN) est inévitable pour les ensembles de toute décomposition de $\mathcal{E}^n$ en une infinité de puissance $< 2^{\aleph_0}$ d'ensembles homogènes, non mesurables et superposables par translation (en particulier, pour la décomposition de S. RUZIEWICZ de [2], si $\mathfrak{m} < 2^{\aleph_0}$). Il existe une décomposition de $\mathcal{E}^n$ en $\mathfrak{m}$ ($\aleph_0 \leq \mathfrak{m} \leqq 2^{\aleph_0}$) ensembles homogènes, saturément non mesurables, superposables par translation et chacun d'eux contenant un ensemble parfait. On donne une démonstration du théorème de I. HALPERIN

de [3]. On donne les analogues descriptifs (dans les termes de la catégorie de Baire) de certains de ces résultats.

2. Par *ensemble homogène non trivial* on comprendra dans la suite un ensemble linéaire, homogène, non dénombrable et autre que le continu linéaire.

LEMME 1. *Un ensemble homogène non trivial est un ensemble frontière partout dense.* (La démonstration est évidente.)

THÉORÈME 1. *Il n'existe aucune décomposition du continu linéaire en un nombre fini >1 d'ensembles disjoints homogènes et superposables par translation.*

DÉMONSTRATION. Soit, par absurde, $\bigcup_{i=1}^{n} E_i$ une telle décomposition. On sait que la famille de tous les ensembles linéaires distincts, superposables par translation avec E_i, est infinie (théorème 1 de [4]). Il existe donc un ensemble F distinct de tout E_i $(i=1, 2, \ldots, n)$ et superposable par translation avec tout E_i. Mais, parce que $\bigcup_{i=1}^{n} E_i$ épuise la droite, il existe un E_j $(1 \leq j \leq n)$ tel que $E_j \cap F \neq 0$ et $F - E_j \neq 0$. D'autre part, du fait que E_j est homogène, il résulte ou bien $E_j \cap F = 0$ ou bien $F = E_j$.

La contradiction obtenue achève la démonstration.

REMARQUE. Le cas particulier suivant du théorème 1 est démontré en [5]: *Il n'existe aucune décomposition du continu linéaire en deux ensembles non vides, disjoints et homogènes.** Nous en donnons ici deux autres démonstrations.

PREMIÈRE DÉMONSTRATION. Admettons qu'une telle décomposition existe. Un au moins des deux ensembles E et F de la décomposition est non trivial. D'après le lemme 1, E et F sont partout denses, donc $f(x)$, la fonction caractéristique de E, est discontinue en chaque point. D'autre part, on voit aisément que $f(x)$ est symétriquement continue en chaque point, c'est-à-dire, pour un x quelconque, on a

$$\lim_{h \to 0} (f(x+h) - f(x-h)) = 0.$$

Mais cela contredit le théorème de [6], qui affirme que, si une fonction est symétriquement continue en chaque point, ses points de discontinuité forment un ensemble de première catégorie.

DEUXIÈME DÉMONSTRATION. En désignant par $D(A)$ l'ensemble des distances de l'ensemble A, on a, en tenant compte que E et F devraient être

* Il est aisé de voir que, s'il existait une telle décomposition, les deux ensembles de la décomposition seraient superposables par translation.

superposables par translation,

$$D(E)=D(F) \tag{1}$$

D'autre part on a l'implication

$$\delta \notin D(E) \Longrightarrow 2\delta \in D(F). \tag{2}$$

Supposons que $0 \in E$; on a donc $D(E) = E$ et, d'après le lemme 1, il existe un $\omega \notin D(E)$. Si l'on a $\omega/2 \in D(E)$, alors, vu l'homogénéité de E, on a aussi $\omega \in D(E)$, ce qui est contradictoire. Si l'on a $\omega/2 \notin D(E)$, alors, d'après (2), on a $\omega \in D(F)$ et, en tenant compte de (1), on a $\omega \in D(E)$, donc de nouveau la même contradiction est établie.

Le théorème 1 donne la réponse négative à la question posée par BOREL sur l'existence d'une décomposition finie de la droite en ensembles homogènes superposables par translation. Mais il est intéressant de savoir si la réponse est aussi négative même quand on renonce à la restriction que les ensembles soient superposables par translation. Nous allons résoudre ce problème, en utilisant le théorème 1.

3. S. BANACH a démontré [7] qu'il existe une fonction d'ensemble non négative, simplement additive, définie pour tout ensemble borné de la droite, telle que pour deux ensembles congruents la fonction prend la même valeur et telle encore que la valeur de la fonction, pour tout ensemble borné mesurable au sens de Lebesgue, est justement la mesure de Lebesgue de cet ensemble.

Nous appelons cette fonction *mesure de Banach* et nous la désignons par ν.

LEMME 2. *Si A est un ensemble homogène non trivial, alors on a* $\nu(A \cap (0,1)) = 0$.

DÉMONSTRATION. On peut admettre que $0 \in A$. Il existe, en tenant compte du lemme 1, un nombre réel positif k_1 tel que $1 > k_1 \notin A$. Désignons par A_1 l'ensemble des points de la forme $x+k_1$ où $x \in A$. A_1 est homogène et $A \cap A_1 = 0$. En vertu du théorème 1, $A \cup A_1$ n'épuise pas la droite. Si $k_2 \notin A \cup A_1$, alors l'ensemble A_2 des points de la forme $x+k_2$, où $x \in A$, est homogène non trivial. En vertu du lemme 1, on peut supposer que $0 < k_2 < 1$. Maintenant nous répétons le même raisonnement. S'il existait un nombre entier positif n, tel que $A \cup A_1 \cup \ldots \cup A_n$ épuise la droite, alors il serait contradictoire au théorème 1. Donc il existe une suite $k_1, k_2, \ldots$ telle que les ensembles $A_1, A_2, \ldots$ sont disjoints et superposables par translation avec A.

Posons $A^* = A \cap (0,1)$ et désignons par A_n^* l'ensemble des points $x+k$ où $x \in A^*$. Si l'on avait $\nu(A^*) = \omega > 0$, alors pour tout n

$$\nu\left(\bigcup_{i=1}^{n} A_i^*\right) = \sum_{i=1}^{n} \nu(A_i^*) = n\,\omega$$

et puisque pour tout n on a $A_n^* \subset (0, 2)$ et en tenant compte que la mesure de Banach est non négative, il résulte, pour tout n, $n\omega \leq 2$, ce qui est absurde. Donc $\nu(A^*) = 0$.

Remarque. Du lemme 2 il résulte que tout ensemble homogène non trivial, mesurable au sens de Lebesgue, est de mesure nulle (ce qui explique pourquoi était il nécessaire que les ensembles homogènes construits par Borel, sans l'axiome du choix, soient de mesure nulle). On peut établir un résultat plus général : *Tout ensemble E homogène non trivial est de mesure intérieure nulle.* En effet, s'il n'était ainsi, l'ensemble $D(E)$ contiendrait un intervalle [8], donc, vu l'homogénéité de E, E aussi contiendrait un intervalle chose impossible à cause du lemme 1.

On a aussi : *Tout ensemble jouissant de la propriété de Baire, contenu dans un ensemble homogène non trivial, est de première catégorie.*

Théorème 2. *Il n'existe aucune décomposition de la droite en un nombre fini > 1 d'ensembles homogènes disjoints et non vides.*

Démonstration. Supposons, par réduction à l'absurde, qu'une telle décomposition $E_1 \cup E_2 \cup \ldots \cup E_n$ existe. Pour tout $1 \leq i \leq n$ tel que E_i est non trivial on a, d'après le lemme 2, $\nu(E_i \cap (0, 1)) = 0$. D'autre part, un E_i trivial est dénombrable, donc de mesure nulle au sens de Lebesgue, donc aussi de mesure nulle au sens de Banach. On a donc aussi $\nu(E_i \cap (0, 1)) = 0$. En vertu de l'additivité simple de ν, on a $\nu((0, 1)) = 0$, ce qui est contradictoire.

Remarque. Le théorème sur l'existence de la mesure de Banach, utilisé dans la démonstration du théorème 2, cesse d'être vrai dans un espace euclidien de dimension supérieure à deux [9]. Cependant, il est remarquable que le théorème 2 est vrai pour un espace euclidien de dimension quelconque, comme le montre le suivant

Corollaire. *Il n'existe aucune décomposition de $\mathcal{E}^n$ en un nombre fini $p > 1$ d'ensembles homogènes, disjoints et non vides.*

Démonstration. Supposons le contraire et soit $\mathcal{E}^n = E_1 \cup E_2 \cup \ldots \cup E_p$ $(p > 1)$ une telle décomposition. Il est facile de voir qu'il existe une droite $D \subset \mathcal{E}^n$ telle que, quel que soit i $(1 \leq i \leq p)$, $E_i \cap D$ n'épuise pas D.

En effet, dans le cas contraire on aurait, pour toute droite D, un certain ensemble E_j $(1 \leq j \leq n)$ tel que $D \subset E_j$. Mais alors, P étant un point quelconque et E_k $(1 \leq k \leq p)$ étant tel que $P \in E_k$, on aurait, pour toute droite D passant par P, $D \subset E_k$, donc $\mathcal{E}^n$ serait épuisé par E_k, contrairement à l'hypothèse $p > 1$.

Remarquons maintenant que chaque $E_i \cap D$ $(1 \leq i \leq p)$ est aussi un ensemble homogène. On obtient donc une décomposition de D en un nombre fini > 1 d'ensembles homogènes disjoints et non vides, chose interdite par le théorème 2.

REMARQUE. Il serait intéressant de trouver, pour le théorème 2 et son corollaire, une démonstration directe (donc qui n'utilise pas la mesure de Banach).

4. Il est aisé de voir que les niveaux d'une fonction additive sont des ensembles homogènes superposables par translation. En tenant compte alors d'un théorème de I. HALPERIN [3], on a la

PROPOSITION H. *Quel que soit le nombre cardinal* $\mathfrak{m}$ *tel que* $\aleph_0 \leq \mathfrak{m} \leq 2^{\aleph_0}$, *il existe une décomposition de la droite en* $\mathfrak{m}$ *ensembles homogènes, non trivials, disjoints, superposables par translation et chacun d'eux rencontrant tout ensemble mesurable E suivant un ensemble de mesure extérieure égale à la mesure de E.*

S. RUZIEWICZ a démontré le théorème suivant [2]: Quel que soit le cardinal $\mathfrak{m}$ tel que $\aleph_0 \leq \mathfrak{m} \leq 2^{\aleph_0}$, il existe une décomposition de la droite en $\mathfrak{m}$ ensembles disjoints, non mesurables et superposables par translation.

Ce théorème est plus faible que celui de I. HALPERIN, qui affirme en plus que chaque ensemble de la décomposition rencontre tout ensemble mesurable E suivant un ensemble de mesure extérieure égale à la mesure de E. La Proposition H contient comme cas particuliers non seulement le théorème de S. RUZIEWICZ, mais aussi ceux de N. LUSIN et W. SIERPINSKI [10], H. HAHN et A. ROSENTHAL [11], C. BURSTIN [12], [13] et O. RINDUNG [14].

Nous allons donner un théorème qui montre, entre autres, que pour $\aleph_0 \leqq \mathfrak{m} < 2^{\aleph_0}$ la décomposition de S. RUZIEWICZ possède toutes les propriétés de la décomposition de I. HALPERIN.

THÉORÈME 3. *Si* $\aleph_0 \leq \mathfrak{m} \leq 2^{\aleph_0}$ *et si* $\bigcup E_\iota$ *est une décomposition de la droite en* $\mathfrak{m}$ *ensembles disjoints, homogènes, superposables par translation et non mesurables au sens de Lebesgue (resp. dépourvus de la propriété de Baire), alors pour tout ensemble mesurable (resp. jouissant de la propriété de Baire) E, on a* $\mu^*(E_\iota \cap E) = \mu(E)$ *(resp. chaque ensemble F jouissant de la propriété de Baire et contenu dans* $E - E_\iota$ *est de première catégorie), quel que soit* ι.

DÉMONSTRATION. En vertu de l'homogénéité des ensembles E_ι, il existent seulement $\mathfrak{m}$ ensembles distincts superposables par translation avec un certain E_ι. Si E_ι est non mesurable, alors un théorème de W. SIERPINSKI (théorème 3 de [4]) affirme que pour tout intervalle I on a $\mu^*(E_\iota \cap I) = \mu(I)$. Cela étant vrai pour tout E_ι, on a pour tout E_ι et pour tout E mesurable $\mu^*(E_\iota \cap E) = \mu(E)$.

Si E_i est dépourvu de la propriété de Baire, alors, en désignant par $E_i(t)$ l'ensemble qui s'obtient de E_i par une translation égale à t, on remarque que, pour une suite $\{t_n\}$ $(n=1, 2, \ldots)$ dense sur la droite, l'ensemble $\bigcup_{n=1}^{\infty} E_i(t_n)$ est de deuxième catégorie de Baire dans chaque intervalle. On parvient ainsi à un analogue descriptif du théorème 3 de [4], analogue qui permet d'accomplir la démonstration de notre théorème dans le cas descriptif aussi.

Un ensemble homogène de la décomposition de I. HALPERIN ne contient aucun ensemble parfait non vide. Il se montre intéressant alors le

THÉORÈME 4. *Si* $\aleph_0 \leq \mathfrak{m} \leq 2^{\aleph_0}$, *il existe une décomposition de la droite en* $\mathfrak{m}$ *ensembles homogènes, disjoints, superposables par translation et tels que chacun d'eux contient un ensemble parfait non vide.*

DÉMONSTRATION. Soit B une base de Hamel qui contient un ensemble parfait P. (L'existence d'une telle base a été démontré par F. B. JONES [15].) Soient P_1 et R_1 deux sousensembles parfaits et disjoints de P. Soit $B = M_1 \cup N_1$, où $P_1 \subset M_1$, $R_1 \subset N_1$, $M_1 \cap N_1 = 0$. On a donc $\overline{\overline{M_1}} = \overline{\overline{N_1}} = 2^{\aleph_0}$. Il existe donc $M \subset M_1$ avec $\overline{\overline{M}} = \mathfrak{m}$.

Posons $N = N_1 \cup (M_1 - M)$. On a $B = M \cup N$. Considérons la fonction

$$f(x) = \begin{cases} x & \text{si } x \in M, \\ 0 & \text{si } x \in N, \end{cases}$$

et définie par le procédé de HAMEL pour $x \notin B$.

Il est aisé de voir que les ensembles de niveau de $f(x)$ fournissent la décomposition cherchée.

REMARQUE. Pour $\mathfrak{m} = \aleph_0$, on a, pour chaque ensemble E_i de la décomposition et pour chaque ensemble mesurable E, $\mu^*(E_i \cap E) = \mu(E)$. Peut-on satisfaire à cette condition, dans le théorème 4, pour $\mathfrak{m} > \aleph_0$? Une réponse affirmative à cette question est donnée par le théorème 5 ci-dessous. Mais d'abord un lemme.

On dit qu'un ensemble A est *rationnellement indépendant* si, pour chaque partie finie $\{a_1, a_2, \ldots, a_n\}$ de A, l'égalité $\sum_{i=1}^{n} r_i a_i = 0$, où r_i sont rationnels, entraîne $r_1 = r_2 = \cdots = r_n = 0$.

LEMME 3. *Il existe un ensemble T parfait, non vide, rationnellement independant, tel que l'ensemble de tous les nombres de la forme $\sum_k r_k x_k$ (somme finie), où r_k sont rationnels et $x_k \in T$, est de mesure nulle.*

Nous allons donner deux démonstrations de ce lemme. La première fournit une construction explicite, tandis que la deuxième prouve seulement l'existence de l'ensemble en cause.

Première démonstration. Soit A_t la suite dont le n-ième terme est $2^n+[n^t]$ $(0<t<\infty)$ où $[x]$ désigne la partie entière de x. Il est aisé de voir que, pour $t_1 \neq t_2$, l'ensemble $A_{t_1} \cap A_{t_2}$ est fini. Posons

$$\alpha_t = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{k!}$$

où $\varepsilon_k = 1$ si $k \in A_t$ et $\varepsilon_k = 0$ si $k \notin A_t$. L'ensemble formé de tous les α_t, pour $0 \leqq t < \infty$, satisfait les conditions du lemme 3.

Deuxième démonstration. Désignons, pour chaque ensemble X, par $R(X)$ l'ensemble de tous les nombres de la forme $\sum_k r_k x_k$ (somme finie) où r_k sont rationnels et $x_k \in X$. D'après un résultat de W. Sierpinski [16], pour chaque X analytique au sens de Lusin—Souslin, $R(X)$ est aussi un ensemble analytique, donc mesurable.

Soit maintenant un ensemble parfait Π contenu dans une base de Hamel H. Π est rationnellement indépendant. D'autre part, Π contient une famille indénombrable $\{\Pi_t\}$ d'ensembles parfaits, disjoints et non vides [17]. Si Π_1 et Π_2 sont deux ensembles parfaits et disjoints contenus dans Π, alors, en tenant compte que $\Pi \subset H$, les ensembles $R(\Pi_1)$ et $R(\Pi_2)$ sont aussi disjoints.

En vertu du résultat de W. Sierpinski, chaque ensemble $R(\Pi_t)$ est mesurable. La famille $\{R(\Pi_t)\}$ étant indénombrable et formée d'ensembles disjoints et mesurables, contient au moins un ensemble $R(\Pi_\zeta)$ de mesure nulle. L'ensemble $T = \Pi_\zeta$ répond aux conditions du lemme.

Théorème 5. *Si $\aleph_0 \leqq \mathfrak{m} \leqq 2^{\aleph_0}$, alors il existe une décomposition $\bigcup_t E_t$ de la droite en $\mathfrak{m}$ ensembles homogènes, disjoints, superposables par translation, tels que chaque E_t contient un ensemble parfait non vide et pour chaque ensemble mesurable E on a $\mu^*(E_t \cap E) = \mu(E)$.*

Démonstration. Soit Ω_0 le plus petit ordinal qui correspond à la puissance du continu. Soit

$$P_1,\ P_2, \ldots,\ P_\xi, \ldots \qquad (\xi < \Omega_0)$$

une suite du type Ω_0, formée par tous les ensembles parfaits linéaires, de mesure positive.

En vertu des propriétés de l'ensemble T du lemme 3, il existe une base de Hamel $\mathcal{H}$ telle que

1° $\mathcal{H} \supset T$;

2° $\mathcal{H}$ contient, de chaque P_ξ, au moins deux points qui n'appartiennent pas à T.

$\mathcal{H}$ peut être écrite sous la forme $S_1 \cup S_2 \cup S_3$ où $S_i \cap S_j = 0$ si $i \neq j$ $(i, j = 1, 2, 3)$ et où $S_1 = T$, S_2 rencontre tout ensemble parfait de mesure positive et S_3 a la puissance $\mathfrak{m}$. Il est visible que pour tout ensemble mesurable E on a $\mu^*(S_2 \cap E) = \mu(E)$.

Désignons par A l'ensemble de tous les nombres de la forme $\sum_k r_k x_k$ (somme finie) où r_k sont rationnels et $x_k \in S_1 \cup S_2$. Désignons par $A + k$ l'ensemble qui s'obtient de A par la translation de longueur k. Si z_t parcourt l'ensemble des nombres de la forme $\sum_k r_k x_k$ (somme finie), où r_k sont rationnels et $x_k \in S_3$, alors $\bigcup_t E_t$, où $E_t = A + z_t$, est une décomposition de la droite qui satisfait toutes les conditions du théorème.

Nous avons remarqué ci-dessus que la *Proposition H* est plus forte que le théorème de S. RUZIEWICZ. Mais du théorème 3 il résulte, au moins dans le cas $\mathfrak{m} < 2^{\aleph_0}$, que cela n'est qu'une apparence. Nous allons montrer maintenant qu'une certaine modification du procédé de S. RUZIEWICZ fournit pour la Proposition H une démonstration semblable à celle donnée par I. HALPERIN.

DÉMONSTRATION DE LA PROPOSITION H. Soit B une base de Hamel qui rencontre tout ensemble parfait. (L'existence d'une telle base a été démontré par C. BURSTIN [13].) On sait [17] qu'un ensemble parfait non vide contient une infinité de puissance $2^{\aleph_0}$ d'ensembles parfaits disjoints et non vides. Il en résulte que B a en commun avec tout ensemble parfait un ensemble de puissance $2^{\aleph_0}$.

Soit Ω_0 le plus petit ordinal qui correspond à la puissance $2^{\aleph_0}$. Rangeons les ensembles parfaits de la droite dans une suite $\{P_\alpha\}$ du type Ω_0. Définissons, par induction, pour chaque α $(1 \leq \alpha < \Omega_0)$ deux éléments $a_\alpha \in P_\alpha$, $b_\alpha \in P_\alpha$ tels que $a_\alpha \in B$, $b_\alpha \in B$ et $a_\alpha \neq a_\beta, b_\beta$, $b_\alpha \neq a_\beta, b_\beta$ pour tout $\beta < \alpha$. Désignons par M_1 l'ensemble des éléments a_α $(1 \leq \alpha < \Omega_0)$ et posons $N_1 = B - M_1$. Il est visible que M_1 et N_1 rencontrent tout ensemble parfait, donc, pour tout E mesurable, on a $\mu^*(M_1 \cap E) = \mu^*(N_1 \cap E) = \mu(E)$. Soit $\mathfrak{m}$ tel que $\aleph_0 \leq \mathfrak{m} \leq 2^{\aleph_0}$ et soit $M \subset M_1$ tel que $\overline{\overline{M}} = \mathfrak{m}$. Considérons la fonction

$$f(x) = \begin{cases} x \text{ si } x \in M, \\ 0 \text{ si } x \in N_1 \cup (M_1 - M), \\ \sum r_i f(x_i) \text{ si } x \notin B, \end{cases}$$

où $x = \sum r_i x_i$ est la représentation de x à l'aide de la base B (r_i sont rationnels, $x_i \in B$).

Les ensembles de niveau de $f(x)$ fournissent la décomposition cherchée.

L'extension à l'espace $\mathcal{E}^n$. Pour l'extension de la Proposition H et des théorèmes 4 et 5 à l'espace $\mathcal{E}^n$ $(n > 1)$, il suffit de remarquer que toute fonc-

tion additive, de n variables, est une somme de n fonctions additives d'une seule variable et que les ensembles de niveau d'une fonction additive de n variables sont homogènes et superposables par translation.

Quant au théorème 3, il est aussi vrai dans $\mathcal{E}^n$, car tout ensemble homogène, non mesurable de $\mathcal{E}^n$ est dense partout dans $\mathcal{E}^n$. Du reste, la démonstration se réduit au cas linéaire.

Remarquons enfin qu'il est facile de voir, en partant d'une base de Hamel dépourvue de la propriété de Baire (la base de Burstin est une telle base), que la démonstration ci-dessus de la Proposition H conduit aussi à une proposition analogue de nature descriptive que nous nous dispensons de formuler.

(Reçu le 29 juillet 1957.)

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ON A DIRECTLY INDECOMPOSABLE ABELIAN GROUP OF POWER GREATER THAN CONTINUUM

By

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(*Presented by* G. HAJÓS)

The aim of this note is to give an example for a torsion free abelian group which is directly indecomposable and whose power exceeds the power $\aleph$ of the continuum. Up to now the existence of such groups has been an open question.[1] The recent works of BOGNÁR [1] and DE GROOT [2] contain examples for directly indecomposable abelian groups until the power of the continuum. The idea of the next example has sprung from a simple combination of a result of DE GROOT with the method of BOGNÁR.

DE GROOT has namely shown [2] that there are $2^{\aleph}$ directly indecomposable groups G_λ such that none of them can be mapped homomorphically onto a non-zero subgroup of another of them. An obvious alteration of his proof shows that there is no loss of generality in assuming that no G_λ contains elements which are divisible by all powers of a fixed prime p.

In each G_λ we pick out an element g_λ not divisible by p and define G as the group generated by the direct sum H of all these G_λ and by all $p^{-1}(g_\lambda+g_\mu)$, $\lambda \neq \mu$.

Now if G_λ is mapped by some homomorphism η into G, then pG_λ is mapped into H and $(pG_\lambda)\eta$ is a subdirect sum of certain subgroups H_μ of G_μ. By a known property of subdirect sums the components are homomorphic images and therefore in our case all H_μ with the exception of H_λ are zero, i. e. $(pG_\lambda)\eta \subseteq G_\lambda$ and hence the image of G_λ under η is a subgroup of G_λ. We conclude that the G_λ are fully invariant subgroups of G, i. e. they are mapped into themselves by every endomorphism of G.

If $G = A+B$ is a direct decomposition of G, then the fully invariant subgroups G_λ are the direct sums of the meets $G_\lambda \cap A$ and $G_\lambda \cap B$, and hence we obtain that any G_λ belongs to either A or B, since the G_λ are directly indecomposable. If G_λ belongs to A and G_μ belongs to B, then consider the element $p^{-1}(g_\lambda+g_\mu) = a+b$ $(a \in A, b \in B)$. It follows that $pa = g_\lambda$ and $pb = g_\mu$, in contradiction to the fact that g_λ and g_μ are not divisible by p.

[1] *Added in proof*, 20 December 1957. In the meantime E. SĄSIADA has published an example, in spirit different from ours; see *Bull. Acad. Pol. Sci. Cl. III*, 5 (1957), pp. 701–703.

We arrive at the result that the group G defined above is directly indecomposable and, clearly, it is of power $2^{\aleph}$.

Let us note that the above method yields $2^{2^{\aleph}}$ non-isomorphic directly indecomposable groups of power $2^{\aleph}$. In fact, we can construct $2^{2^{\aleph}}$ different subsets, of power $2^{\aleph}$, of the set of all G_λ and for each subset of the G_λ we may apply the above construction in order to obtain groups G. These will be non-isomorphic.

As an application we mention that the group H constructed above disproves a conjecture made by SZELE and SZENDREI [3]. According to this conjecture there exist no groups of power greater than $\aleph$ whose endomorphism ring is commutative. The endomorphism ring of H is the direct sum of the endomorphism rings of the fully invariant subgroups G_λ and it is easily seen that a G_λ admits only multiplications by rationals as endomorphisms, i. e. the endomorphism rings of the G_λ are subrings of the rational number field, and therefore they are commutative.

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ON A PROBLEM OF M. H. STONE

By
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(*Presented by* L. RÉDEI)

Pseudo-complemented lattices form an important class of (distributive) lattices. Topological distributive lattices, the lattice of all ideals of a distributive lattice with zero element, the lattice of all congruence relations of an arbitrary lattice are all pseudo-complemented. It is clear that the Boolean algebras have the same property.

Thus we might consider the distributive pseudo-complemented lattices in which $a^* \cup a^{**} = 1$ holds for all a as an immediate generalization of the Boolean algebras. The investigation of this type of lattices was proposed by M. H. STONE (it is G. BIRKHOFF's problem 70, see[1] [1], p. 149):

What is the most general pseudo-complemented distributive lattice in which $a^* \cup a^{**} = 1$ identically?

In this paper we get two solutions of STONE's problem. After this we deal with a related question.

§ 1. Preliminaries

We begin by giving some definitions.

DEFINITION 1. The (distributive) lattice L is called *pseudo-complemented* if it has a zero element and for any element a of L there exists an element a^* of L such that $a \cap x = 0$ if and only if $x \leq a^*$. The element a^* is called the pseudo-complement of a.

DEFINITION 2. A lattice L is said to be a *Stone lattice* if it is a pseudo-complemented distributive lattice with unit element in which $a^* \cup a^{**} = 1$ for each element a of L.

DEFINITION 3. We shall call the lattice L *relative Stone lattice* if every closed interval of L is a Stone lattice.

REMARK. We mention the fact that a relative Stone lattice is a Stone lattice if and only if it has zero and unit elements. A Stone lattice is not a

[1] Numbers in brackets refer to the Bibliography given at the end of the paper.

relative Stone lattice in general, e. g. if we define a new zero and unit element for an arbitrary Boolean algebra, then this lattice is a Stone lattice which is not a relative Stone lattice.

DEFINITION 4. Let L be a lattice with zero element. The element b is said to be a *semi-complement*[2] of the element a if $a \cap b = 0$. The lattice is called *dense* if $a \cap b = 0$ implies that a or b is 0.

Now we recall a few facts on which the sequel depends.

LEMMA 1 (M. H. STONE's theorem). *Let L be a distributive lattice, I an ideal and D a dual ideal of L such that I and D are disjoint. Any maximal ideal P, for which $P \supseteq I$ further P and D are disjoint, is prime.*

The proof is well known (see [3] too).

LEMMA 2. *Let L be a distributive lattice with zero element and P a prime ideal of L. There exists a minimal prime ideal Q with $Q \subseteq P$.*

PROOF. If P is a prime ideal, then[3] $L-P$ is a dual prime ideal (see [1], p. 141). A maximal dual ideal Q which contains $L-P$ and $0 \notin Q$ is a dual prime ideal (Lemma 1), that is, $L-Q$ is a minimal prime ideal in P.

LEMMA 3. *If in a distributive lattice the meet and the join of two ideals are principal ideals, then the given ideals are also principal ideals.*

This was proved in [3].

LEMMA 4. *Under any lattice homomorphism, the complete inverse image of a prime ideal is again a prime ideal.*

This result may be found in [3].

LEMMA 5. *Let L be a distributive lattice and D a dual ideal of L. There exists a minimal congruence relation on L under which D is a congruence class. Under this congruence relation $a \leq b$ and $a \equiv b$ are equivalent to the condition that there exists an element $d \in D$ with $b \cap d = a$.*

This is equivalent to Corollary 4 of Theorem 2 of [2].

§ 2. Characterizations of Stone lattices

The main result of this paper is

THEOREM 1. *Let L be a distributive pseudo-complemented lattice with unit element. Then L is a Stone lattice if and only if the lattice-theoretical join of any two distinct minimal prime ideals of L is L.*

[2] This notion is due to G. SZÁSZ [4].

[3] $P-Q$ denotes the set-theoretical difference and later on $P+Q$ the set-theoretical sum.

PROOF. Let L be a distributive lattice with zero and unit elements in which the join of any two distinct minimal prime ideals is L. We must prove that $a^* \cup a^{**} = 1$ for all $a \in L$. (We have supposed that a^* exists.)

We suppose that there exists an element a for which $a^* \cup a^{**} \neq 1$. By Lemma 1 there exists a dual prime ideal P with $a^* \cup a^{**} \notin P$. Let us consider the minimal congruence relation Θ on L under which P is a congruence class. We assert that in the factor lattice L/Θ the join of any two distinct minimal prime ideals is the whole lattice.

Let $\bar{Q}$ and $\bar{R}$ be minimal prime ideals of L/Θ and Q, R their complete inverse images. By Lemma 4, Q and R are prime ideals; we prove that they are minimal ones. Indeed, if $Q_1 \subset Q$ (Q_1 is a prime ideal), then the homomorphic image of Q_1 and Q coincide, hence for arbitrary $q_1 \in Q_1$ and for some $q \in Q - Q_1$ the relation $q \equiv q_1 (\Theta)$ is valid. We may suppose $q_1 < q$ so that by Lemma 5 there exists a $p \in P$ which satisfies $q \cap p = q_1$. But $p \notin Q_1$, for in case $p \in Q_1$, p would be an element common to Q_1 and to $P = \bar{1}$ which is a contradiction. Thus we get that p and q are not elements of the prime ideal Q_1, nevertheless $p \cap q \in Q_1$. This contradiction proves our assertion.

We get that in L/Θ the join of any two distinct minimal prime ideals is the whole lattice. Now we intend to show that in L/Θ there exists only one minimal prime ideal: $(\bar{0}]$.

The unit element of L/Θ is join-irreducible, for in case $\bar{x} \cup \bar{y} = \bar{1}$ and $\bar{x}, \bar{y} \neq \bar{1}$, $x \cup y \in P$ but $x, y \notin P$ which is absurd, because P is a dual prime ideal. Consequently, $\bar{L}$ contains only one minimal prime ideal, for if in $\bar{L}$ there were two minimal prime ideals, then the join of these would be the whole lattice, i. e $\bar{1}$ would be join-reducible which is impossible. Finally, let $\bar{S}$ be any minimal prime ideal of L/Θ and $\bar{S} \neq (\bar{0}]$. We choose an $\bar{a} \in \bar{S}$. By the *Duality Principle* and by Lemma 1 there exists a prime ideal which does not contain $[\bar{a})$. This prime ideal, by Lemma 2, contains a minimal one which is obviously different from $\bar{S}$.

We have supposed that $a^* \cup a^{**} < 1$, consequently $0 < a^*$. We assert that $a \neq 0 (\Theta)$. Indeed, in case $a \equiv 0 (\Theta)$ it follows the existence of a $p \in P$ (Lemma 5) such that $a \cap p = 0$, i. e. $p \leq a^*$, hence p is an element common to P and to $(a^* \cup a^{**}]$, which is a contradiction. Similarly, $a^* \neq 0 (\Theta)$.

We get that in L/Θ $\bar{0} < \bar{a}$ and $\bar{0} < \bar{a}^*$, yet $\bar{a} \cap \bar{a}^* = \bar{0}$, in contradiction to the fact that $(\bar{0}]$ is a prime ideal.

Thus we have proved that in a pseudo-complemented lattice with unit element, if the join of any two distinct minimal prime ideals is the whole lattice, then $a^* \cup a^{**} = 1$ for all elements a of the lattice, i. e. it is a Stone lattice.

Conversely, let L be a Stone lattice, T and U distinct minimal prime ideals of L. $L-T$ and $L-U$ are maximal dual prime ideals, consequently, there exist $a \in L-U$ and $b \in L-T$ with $a \cap b = 0$. Obviously $a \in T-U$ and $b \in U-T$ is valid too, since otherwise a and b would be in the same dual prime ideal $L-T$ or $L-U$ which is impossible in view of $a \cap b = 0$. $a \in T$, so $a^* \in U-T$ and $a^{**} \in T-U$, hence from $a^* \cup a^{**} = 1$ it follows $T \cup U = (1]$.

Another — almost obvious — characterization of Stone lattices is the following

THEOREM 2. *A distributive lattice L with 0 and 1 is a Stone lattice if and only if for all $a \in L$ the ideal formed by the semi-complements of a is a direct factor of L.*

PROOF. Let L be a Stone lattice. The ideal formed by the semi-complements of a is clearly $(a^*]$, hence $(a^*] \cap (a^{**}] = (0]$ and $(a^*] \cup (a^{**}] = (1]$; thus $(a^*]$ is indeed[4] a direct factor of L.

Conversely, let I be the ideal formed by the semi-complements of an element a. If I is a direct factor, then there exists an ideal J with $I \cap J = (0]$ and $I \cup J = (1]$. By Lemma 3, it follows that I and J are principal ideals, moreover the generating elements are a^* and a^{**}. Thus the proof of Theorem 2 is complete.

If L is a Stone lattice, then it is either dense or there exists an element a $(0 < a < 1)$ such that $a^* > 0$. But in the latter case, by Theorem 2, L is directly factorisable. Thus, an immediate consequence of Theorem 2 is the following

COROLLARY. *A finite distributive lattice L is a Stone lattice if and only if it is the direct product of dense lattices.*

§ 3. Relative Stone lattices

The following theorem is analogous to Theorem 1 in case of relative Stone lattices:

THEOREM 3. *Let L be a distributive lattice in which every closed interval (as a sublattice) is a pseudo-complemented lattice. L is a relative Stone lattice if and only if in L for any pair of prime ideals P and Q, of which neither contains the other, $P \cup Q = L$ is valid.*

[4] It is well known that I is a direct factor of the distributive lattice L with zero and unit elements if and only if there exists an element a such that $I = (a]$ and a has a complement.

Proof. Let us suppose that although L is a relative Stone lattice, there exists a pair of prime ideals P and Q such that $P \cup Q \subset L$, but neither $P \subseteq Q$ nor $Q \subseteq P$. We choose $a \in L-(P \cup Q)$, $b \in P-Q$ and $c \in Q-P$. By the hypothesis the interval $[b \cap c, a \cup b \cup c]$ as a sublattice is a Stone lattice. Hence, in this interval b has a pseudo-complement b^*. b^* is necessarily in $Q-P$, and $b^{**} \in P-Q$; in consequence of this fact $b^* \cup b^{**} = a \cup b \cup c$, i. e. $a \cup b \cup c \in P \cup Q$, but we have supposed $a \notin P \cup Q$. Thus the proof of the necessity of the conditions is completed.

On the other hand, assume that for any pair of prime ideals P and Q of this lattice L, none of them containing the other, $P \cup Q = L$ is valid. Now, let us consider an interval $[a, b]$ of L and two minimal prime ideals P' and Q' of $[a, b]$. There exists a pair of prime ideals P, Q of L with the property that $P \cap [a, b] = P'$ and $Q \cap [a, b] = Q'$. Indeed (see Lemma 1), let P be a maximal ideal which contains the ideal of L generated by P' and is disjoint from the dual ideal of L generated by $[a, b]-P'$; Q may be defined in a similar way. Obviously, none of P and Q contains the other, hence, by our assumption $P \cup Q = L$. It follows that $P' \cup Q' = [a, b]$. Applying Theorem 1, we get that the interval $[a, b]$ is a Stone lattice, consequently, L is a relative Stone lattice.

It is easy to characterize the relative Stone lattices if we apply the following

Theorem 4. *If every closed interval of a distributive lattice L is pseudo-complemented and if L has no homomorphic image isomorphic to the lattice of Fig. 1, then L is a relative Stone lattice.*

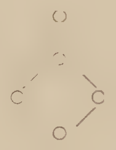

Fig. 1

Proof. We must prove that if the distributive lattice L, in which every closed interval is as a sublattice pseudo-complemented, is not a relative Stone lattice, then it has a homomorphic image isomorphic to the lattice of Fig. 1. By Theorem 3, if L is not a relative Stone lattice, then it has a pair of prime ideals P, Q such that $P \not\subseteq Q$, $P \not\supseteq Q$ and $P \cup Q \subset L$. By Lemma 1, there exists in L a prime ideal R with $P \cup Q \subseteq R$. We define a congruence relation Θ on L as follows: let $H_1 = P \cap Q$, $H_2 = P-Q$, $H_3 = Q-P$, $H_4 = L-R$, $H_5 = R-(P \cup Q)$ and let $x \equiv y$ (Θ) if and only if for some $i (= 1, 2, \ldots, 5)$ x and y are both in H_i. It is routine to check that Θ is a congruence rela-

tion. It is obvious that $L\,\Theta$ is isomorphic to the lattice of Fig. 1, what was to be proved.

Finally, we mention the problem whether Theorem 1 is valid if pseudo-complementedness is not assumed.

The interest of this problem lies in the fact that if every minimal prime ideal is a maximal one, then the assertion is true, see e. g. [3].

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ON THE THEORY OF DIOPHANTINE APPROXIMATIONS. I[1]
(ON A PROBLEM OF A. OSTROWSKI)

By

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(*Presented by* A. RÉNYI)

In what follows we denote by $\langle x \rangle$ the fractional part of the positive x, and by α any number with $0<\alpha<1$. As well known,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle n\alpha \rangle = \frac{1}{2}.$$

We put for an α

$$\sum_{n=1}^{N} \langle n\alpha \rangle - \frac{N}{2} = C_\alpha(N).$$

As A. OSTROWSKI [1][2] has shown, for any irrational α the quantity $C_\alpha(N)$ is unbounded. In the same paper he raised the question whether or not $C_\alpha(N)$ can for an appropriate α be *onesidedly* bounded. In this paper we are going to give to this question an affirmative answer, i. e. to prove the following

THEOREM. *There is an irrational α and a constant C such that for $N=1, 2, \ldots$ the inequality*

$$C_\alpha(N) > C$$

holds.

A slight modification of our construction gives at the same time the existence of a set of such α's having the power of the continuum.

As A. RÉNYI remarked, the constant C of our theorem cannot be ≥ 0; a slight modification of our construction would show that for C we could prescribe any negative number. We shall omit this modification.

In the proof we start from a geometrical interpretation of continued fractions which is applicable also to other questions of diophantine approximations. So I intend to return in this sequence of papers to a theorem of A. KHINTCHINE, i. e. to the lower estimation of

$$\sup_{\beta} \inf_{\substack{x \geq 0 \\ y,\, x \text{ integers}}} x|\alpha x + \beta - y|,$$

[1] The results of this sequence of papers were contained in my dissertation, defended in June 1957, and I lectured on some parts of it in Lublin and Lodz in September 1956.

[2] The numbers in brackets refer to the Bibliography given at the end of the paper.

investigated previously by A. KHINTCHINE [2], S. FUKASAWA [3], H. DAVENPORT [4] and A. V. PRASAD [5], and to the upper estimation of

$$\inf_{\alpha} \inf_{\substack{x \neq 0 \\ x,\, y \text{ integers}}} x|\alpha x+\beta-y|,$$

investigated by J. W. S. CASSELS [6]. Further applications I shall publish elsewhere.

In § 1 of this paper we give this geometrical interpretation of continued fractions and announce some lemmas whose proof will be postponed owing to their general character to the Appendix. In § 2 we deduce from it an "exact formula" for $\sum_{n=1}^{N} \langle n\alpha \rangle$ (Main Lemma). In § 3 we prove the announced theorem.

§ 1

Starting from a fixed point O of the periphery of a circle E with unity periphery we put up in positive direction the arc with the length α $(0<\alpha<1)$ once, twice, ..., n-times, ... The endpoints of these arcs we shall call the "$n\alpha$-points" $(n=1, 2, \ldots)$. We need the following

DEFINITION. We call the $s\alpha$-point "adjacent to O", and the corresponding s an "adjacent multiplum of O", if no $n\alpha$-point with $0<n<s$ is contained in one of the two closed arcs determined by O and the $s\alpha$-point.

So we obtained to our fixed α a sequence

$$0=s_0,\ 1=s_1<s_2<s_3<\cdots \tag{1.1}$$

of adjacent multipla; we shall denote the "empty" open arc corresponding to the $s_\nu\alpha$-point by Δ_ν, the length of this arc by δ_ν. We shall use also the directed "empty" arc between O and the $s_\nu\alpha$-point, the sign of its length being positive and negative, respectively, according to the direction in which the arc Δ_ν starts from O. This length with sign we shall denote by δ_ν.

Particularly important are those s_ν-multipla from (1. 1) for which δ_ν and $\delta_{\nu+1}$ have different signs. We shall call these $s_{\nu_1}, s_{\nu_2}, \ldots, s_{\nu_k}, \ldots$, forming a subsequence of the sequence $s_1, s_2, \ldots, s_\nu, \ldots$, the "jumping-multipla" and denote them by

$$q_1, q_2, \ldots, q_k = s_{\nu_k}, \ldots. \tag{1, 2}$$

In the case when $\frac{1}{2}<\alpha<1$, the definition needs an additional remark, it is suitable to define $q_2=q_1=1$. If $k>1$, then $q_{k+1}>q_k$. The corresponding quantities δ_{ν_k} and δ_{ν_k} we shall denote simply by d_k and d_k, respectively.

We define $\bar{d}_0 = 1$. Next we define

$$a_k = \left[\frac{\bar{d}_{k-1}}{\bar{d}_k}\right] \qquad (k = 1, 2, \ldots). \tag{1.3}$$

LEMMA I. *If the $s_\nu\alpha$-point is adjacent to O and from the opposite side of O the nearest to O among the α-, 2α-, ..., $s_\nu\alpha$-points is the $s_{\nu-l}\alpha$-point (l positive integer), then we have*

$$s_{\nu+1} = s_\nu + s_{\nu-l}, \tag{1.4}$$

$$\delta_{\nu+1} = \delta_\nu + \delta_{\nu-l}. \tag{1.5}$$

The geometrical meaning of Lemma I is that one obtains the arc $\Delta_{\nu+1}$ from the arcs Δ_ν and $\Delta_{\nu-l}$ in the following way: considering the larger of Δ_ν and $\Delta_{\nu-l}$, from its endpoint different from the point O we draw back the smaller of the arcs Δ_ν and $\Delta_{\nu-l}$. This remark will be often used explicitly or implicitly.

LEMMA II. *We have for the above-defined quantities the recursive formulae for $k = 1, 2, \ldots$:*

$$q_{k+1} = q_{k-1} + a_k q_k, \tag{1.6}$$

$$d_{k+1} = d_{k-1} + a_k d_k, \qquad \bar{d}_{k+1} = \bar{d}_{k-1} - a_k \bar{d}_k, \tag{1.7}$$

$$\left.\begin{array}{l} s_{\nu_k+r} = q_{k-1} + r q_k, \quad (1.8) \\ \delta_{\nu_k+r} = d_{k-1} + r d_k, \qquad \bar{\delta}_{\nu_k+r} = \bar{d}_{k-1} - r \bar{d}_k \quad (1.9) \end{array}\right\} \quad (0 < r < a_k),$$

$$q_{k+1}\bar{d}_k + q_k \bar{d}_{k+1} = 1. \tag{1.10}$$

LEMMA III. *If the positive integer n is not an s from* (1.1), *then there exists an s_ν with $s_\nu < n$ and*

$$s_\nu \bar{\delta}_\nu < n t_n \tag{1.11}$$

where

$$t_n = \min(\langle n\alpha \rangle, 1 - \langle n\alpha \rangle).$$

Further, if s_ν is not a q_k from (1.2), *then there is a $q_k < s_\nu$ such that*

$$q_k < s_\nu < q_{k+1}$$

and

$$q_{k+1} d_{k+1} < s_\nu \delta_\nu. \tag{1.12}$$

The proof of these lemmas will follow in the Appendix. This shows that the s_ν-multipla in (1.1) are identical with the set of all denominators and by-denominators (Neben-Nenner) of the convergents of the continued fraction of α, whereas the q-multipla in (1.2) are the denominators of its convergents and the a_k's its digits.[3]

[3] In this paper we use the term "digit" instead of the usual "partial quotient".

Moreover we shall need the following special

Lemma IV. *An arbitrary positive integer N can be represented in the form*

$$N = s_{\mu_1} + s_{\mu_2} + \cdots + s_{\mu_k} \tag{1.13}$$

where the s_{μ_j}-numbers are the adjacent multipla of an arbitrarily prescribed irrational α in the sense of (1.1), *further*

$$\mu_1 > \mu_2 > \cdots > \mu_k$$

and[4]

$$n_j \equiv N - \sum_{i=1}^{j} s_{\mu_i} < s_{\mu_j - 1} \qquad (j = 1, 2, \ldots, k). \tag{1.14}$$

Proof. Is N one of our s_ν-numbers, we have nothing to prove. If not, then there is an index μ_1 with

$$s_{\mu_1} < N < s_{\mu_1+1}.$$

Owing to (1.4) we have

$$n_1 = N - s_{\mu_1} < s_{\mu_1+1} - s_{\mu_1} = s_{\mu_1 - l} \leqq s_{\mu_1 - 1}.$$

Next there is an index μ_2 with $\mu_2 < \mu_1$ and

$$s_{\mu_2} \leqq n_1 < s_{\mu_2+1}.$$

Again we have, using (1.4),

$$n_2 = n_1 - s_{\mu_2} < s_{\mu_2+1} - s_{\mu_2} \leqq s_{\mu_2 - 1}$$

and this process is obviously finished after a finite number of steps.

§ 2

Let N be a positive integer, α a positive irrational number and we represent N in the form (1.13). Then we assert the following

Main Lemma.[5] *With the notation of § 1 and the representation* (1.13) *the formula*

$$\begin{aligned} C_\alpha(N) \equiv \sum_{n=1}^{N} \langle n\alpha \rangle - \frac{N}{2} = \left(\bar{\delta}_{\mu_k} \frac{s_{\mu_k} + 1}{2} - \frac{1}{2} \right) \operatorname{sign} \delta_{\mu_k} + \\ + \sum_{j=1}^{k-1} \left\{ \bar{\delta}_{\mu_j} \left(\frac{s_{\mu_j} + 1}{2} + s_{\mu_{j+1}} + \cdots + s_{\mu_k} \right) - \frac{1}{2} \right\} \operatorname{sign} \delta_{\mu_j} \end{aligned} \tag{2.1}$$

holds.

[4] In the case when $\mu_k = 1$, the last inequality for n_k must be dropped.

[5] An exact formula occurs also in Ostrowski's paper [1]. His formula contains only the denominators of the convergents of the continued fraction of α.

For the proof we shall need the following lemmas:

LEMMA V. *The Main Lemma is true in case of $k=1$, i. e.*

$$\sum_{n=1}^{s_\mu} \langle n\alpha \rangle = \frac{s_\mu}{2} + \left\{ \delta_\mu \frac{s_\mu+1}{2} - \frac{1}{2} \right\} \operatorname{sign} \delta_\mu .$$

PROOF. The $n\alpha$-points ($n=0, 1, 2, \ldots, s_\mu$) divide the periphery of the circle E into $(s_\mu+1)$ disjunct arcs; starting from the point O in positive direction we denote the length of these arcs by $t_0, t_1, \ldots, t_{s_\mu}$, respectively. Since the arcs with the length $\langle n\alpha \rangle$ put up on E from O in positive direction ($n=0, 1, \ldots, s_\mu$) cover the arc with the length t_l ($l=0, 1, \ldots, s_\mu$) obviously $(s_\mu - l)$-times, we have on the one hand

$$\sum_{n=1}^{s_\mu} \langle n\alpha \rangle = \sum_{l=0}^{s_\mu} (s_\mu - l) t_l . \tag{2. 2}$$

On the other hand, we can determine the sum on the left side putting up the arcs $\alpha, 2\alpha, \ldots, s_\mu \alpha$ in the negative direction, starting from the $s_\mu\alpha$-point. These points in their totality coincide obviously with the $n\alpha$-points ($n=0, 1, \ldots, s_\mu$). Thus now the $s_\mu\alpha$-point plays the role of O and we have to sum the distances of our points from the $s_\mu\alpha$-point.

Case I. $\delta_\mu > 0$ (i. e. $\delta_\mu = t_0$). Then expressing our sum again by means of the t_l's we obviously get

$$\sum_{n=1}^{s_\mu} \langle n\alpha \rangle = s_\mu t_0 + (s_\mu - 1) t_{s_\mu} + (s_\mu - 2) t_{s_\mu - 1} + \cdots + 2t_3 + t_2 . \tag{2. 3}$$

Adding (2. 2) and (2. 3) we obtain

$$\sum_{n=1}^{s_\mu} \langle n\alpha \rangle = s_\mu t_0 + \frac{s_\mu - 1}{2} (t_1 + t_2 + \cdots + t_{s_\mu}). \tag{2. 4}$$

Since

$$t_0 + t_1 + \cdots + t_{s_\mu} = 1,$$

(2. 4) gives

$$\sum_{n=1}^{s_\mu} \langle n\alpha \rangle = \frac{1}{2} (s_\mu - 1 + (s_\mu + 1) t_0) =$$

$$= \frac{s_\mu}{2} + \left\{ \bar{\delta}_\mu \frac{s_\mu+1}{2} - \frac{1}{2} \right\} = \frac{s_\mu}{2} + \left\{ \bar{\delta}_\mu \frac{s_\mu+1}{2} - \frac{1}{2} \right\} \operatorname{sign} \delta_\mu .$$

Case II. $\delta_\mu < 0$ (i. e. $\delta_\mu = -t_{s_\mu}$). Then the identity corresponding to (2. 3) is

$$\sum_{n=1}^{s_\mu} \langle n\alpha \rangle = s_\mu t_{s_\mu - 1} + (s_\mu - 1) t_{s_\mu - 2} + \cdots + 2t_1 + t_0 . \tag{2. 5}$$

Adding (2.2) and (2.5) we obtain

$$\sum_{n=1}^{s_\mu} \langle n\alpha \rangle = \frac{s_\mu+1}{2}(t_0+\cdots+t_{s_\mu-1}) = \frac{s_\mu+1}{2}(1-t_{s_\mu}) = \tag{2.6}$$
$$= \frac{s_\mu}{2} + \left\{\delta_\mu \frac{s_\mu+1}{2} + \frac{1}{2}\right\} = \frac{s_\mu}{2} + \left\{\bar{\delta}_\mu \frac{s_\mu+1}{2} - \frac{1}{2}\right\} \operatorname{sign} \delta_\mu .$$

Further we need the simpler

LEMMA VI. *Let m, S be positive integers and let us consider the $(m+j)\alpha$-points $(j=1,2,\ldots,S)$. If one of the arcs determined by O and the $m\alpha$-point is empty and the directed length of this empty arc is $d(m)$, then we have*

$$\sum_{j=1}^{S} \langle (m+j)\alpha \rangle = \sum_{j=1}^{S} \langle j\alpha \rangle + Sd(m).$$

PROOF. The directed distance from the $j\alpha$-point to the $(m+j)\alpha$-point on the circle is the same as between O and the $m\alpha$-point, i. e.

$$\langle (m+j)\alpha \rangle - \langle j\alpha \rangle = d(m)$$

from which summation for $j=1,2,\ldots,S$ already proves the lemma.

Finally we prove the

LEMMA VII. *Using the representation* (1.13) *it holds for $j=1,2,\ldots,k$ that one of the two arcs of the circle E determined by O and the $(s_{\mu_1}+\cdots+s_{\mu_j})\alpha$-point does not contain any $n\alpha$-point whenever*

$$s_{\mu_1}+\cdots+s_{\mu_j} < n \leqq s_{\mu_1}+\cdots+s_{\mu_{j+1}}. \tag{2.7}$$

PROOF. From the point O we can reach the $(s_{\mu_1}+s_{\mu_2}+\cdots+s_{\mu_j})\alpha$-point starting from O going first to the $s_{\mu_1}\alpha$-point along the arc J_{μ_1}, then from the $s_{\mu_1}\alpha$-point to the $(s_{\mu_1}+s_{\mu_2})\alpha$-point along the arc with the directed length δ_{μ_2}, and so forth, and finally from the $(s_{\mu_1}+\cdots+s_{\mu_{j-1}})\alpha$-point to the $(s_{\mu_1}+\cdots+s_{\mu_j})\alpha$-point along the arc with the directed length δ_{μ_j}. We shall prove our lemma a fortiori by showing that for the n's in (2.7) no $n\alpha$-points lie in these arcs with the directed length $\delta_{\mu_1}, \delta_{\mu_2}, \ldots, \delta_{\mu_j}$. First of all from (2.7) it follows that for $i=1,2,\ldots,j$

$$n > s_{\mu_1}+\cdots+s_{\mu_i}. \tag{2.8}$$

If for an n the $n\alpha$-point would lie on the above-mentioned arc with the directed length $\delta_{\mu_{i+1}}$, then the ordering of the points

$$(s_{\mu_1}+\cdots+s_{\mu_i})\alpha,\ n\alpha,\ (s_{\mu_1}+\cdots+s_{\mu_{i+1}})\alpha$$

would be the same as the ordering of the points

$$O,\ (n-s_{\mu_1}-\cdots-s_{\mu_i})\alpha,\ s_{\mu_{i+1}}\alpha.$$

Taking into account (2. 8) and the definition of $s_{\mu_{i+1}}$ it would follow

$$n - s_{\mu_1} - \cdots - s_{\mu_i} \geqq s_{\mu_{i+1}+1},$$

i. e.

$$n \geqq s_{\mu_1} + \cdots + s_{\mu_i} + s_{\mu_{i+1}+1}.$$

But owing to Lemma IV

$$s_{\mu_{i+1}+1} > s_{\mu_{i+1}} + \cdots + s_{\mu_k},$$

i. e. $n > N$ would follow, which is a contradiction.

From the above lemmas the proof of the Main Lemma can be completed as follows. We write

$$\sum_{n=1}^{N} \langle n\alpha \rangle = \sum_{n=1}^{s_{\mu_1}} \langle n\alpha \rangle + \sum_{n=s_{\mu_1}+1}^{s_{\mu_1}+s_{\mu_2}} \langle n\alpha \rangle + \cdots + \sum_{n=s_{\mu_1}+\cdots+s_{\mu_{k-1}}+1}^{s_{\mu_1}+\cdots+s_{\mu_k}} \langle n\alpha \rangle.$$

Owing to Lemma VII, Lemma VI is applicable; using also Lemma V we obtain

$$(2.9)\qquad \begin{aligned} \sum_{n=1}^{N} \langle n\alpha \rangle = {} & \frac{s_{\mu_1}}{2} + \left\{ \bar{\delta}_{\mu_1} \frac{s_{\mu_1}+1}{2} - \frac{1}{2} \right\} \operatorname{sign} \delta_{\mu_1} + \\ & + \frac{s_{\mu_2}}{2} + \left\{ \bar{\delta}_{\mu_2} \frac{s_{\mu_2}+1}{2} - \frac{1}{2} \right\} \operatorname{sign} \delta_{\mu_2} + s_{\mu_2} d(s_{\mu_1}) + \\ & \vdots \\ & + \frac{s_{\mu_k}}{2} + \left\{ \bar{\delta}_{\mu_k} \frac{s_{\mu_k}+1}{2} - \frac{1}{2} \right\} \operatorname{sign} \delta_{\mu_k} + s_{\mu_k} d(s_{\mu_1} + \cdots + s_{\mu_k}). \end{aligned}$$

From what has been said in the proof of Lemma VII it follows

$$d(s_{\mu_1} + \cdots + s_{\mu_j}) = \delta_{\mu_1} + \cdots + \delta_{\mu_j}.$$

Putting it into (2. 9) the proof of the Main Lemma is complete.

§ 3

We shall prove the announced theorem. We use again the representation (1. 13) of N and divide the s_{μ_j}'s into two classes according to the sign of the corresponding δ_{μ_j}. Let

$$(3.1)\qquad \begin{aligned} s_{\mu_j} &= s'_{\mu_j}, & \delta_{\mu_j} &= \delta'_{\mu_j} \quad \text{for} \quad \delta_{\mu_j} > 0, \\ s_{\mu_j} &= s''_{\mu_j}, & \delta_{\mu_j} &= \delta''_{\mu_j} \quad \text{for} \quad \delta_{\mu_j} < 0. \end{aligned}$$

Introducing this notation in the Main Lemma we get

$$(3.2)\qquad \begin{aligned} C_\alpha(N) = {} & \sum_j \left\{ \delta'_{\mu_j} \left(\frac{s'_{\mu_j}+1}{2} + \sum_{s_{\mu_l} < s'_{\mu_j}} s_{\mu_l} \right) - \frac{1}{2} \right\} + \\ & + \sum_j \left\{ \frac{1}{2} - \delta''_{\mu_j} \left(\frac{s''_{\mu_j}+1}{2} + \sum_{s_{\mu_l} < s''_{\mu_j}} s_{\mu_l} \right) \right\} = \Sigma_1 + \Sigma_2. \end{aligned}$$

Omitting from Σ_1 the positive terms $\sum\limits_{s_{\mu_l}<s_{\mu_j}} s_{\mu_l}$ and taking into account that from Lemma IV

$$\sum_{s_{\mu_l}<s''_{\mu_j}} s_{\mu_l} < s''_{\mu_j},$$

we obtain from (3. 2)

$$C_\alpha(N) > \frac{1}{2}\sum_j (\bar{\delta}'_{\mu_j} s'_{\mu_j} - 1) - \frac{3}{2}\sum_j \delta''_{\mu_j} s''_{\mu_j} = \Sigma'_1 + \Sigma'_2. \tag{3. 3}$$

This suggests as a guide for the choice of α that for the s'_{μ_j}-multipla we should have $s'_{\mu_j}\bar{\delta}'_{\mu_j} \sim 1$ and, on the other hand, the products $s''_{\mu_j}\delta''_{\mu_j}$ should be small, i. e. O should be approached "badly" from the positive side and well from the negative one.

The actual construction of such an α can be performed as follows. Denoting the digits of the continued fraction of an α by $a_1, a_2, \ldots$,

$$\alpha = \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+\cdots},$$

we define

$$\left.\begin{aligned} a_{2k-1} &= 1, \quad (3.4)\\ a_{2k} &= k^3 \quad (3.5) \end{aligned}\right\} \qquad (k=1,2,\ldots);$$

owing to $a_1 = 1$ we have $\frac{1}{2} < \alpha < 1$ and owing to the additional remark on p. 462

$$q_1 = q_2 = 1. \tag{3. 6}$$

The formulae (1. 6) and (1. 7) give

$$q_{2k} = q_{2k-1} + q_{2k-2}, \tag{3. 7}$$

$$q_{2k+1} = q_{2k-1} + k^3 q_{2k}, \tag{3. 8}$$

$$\bar{d}_{2k} = \bar{d}_{2k+1} + \bar{d}_{2k+2}, \tag{3. 9}$$

$$\bar{d}_{2k+1} = k^3 \bar{d}_{2k+2} + \bar{d}_{2k+3}. \tag{3. 10}$$

From (1. 10) and (3. 8) we obtain

$$q_{2k}\bar{d}_{2k} = \cfrac{1}{\cfrac{q_{2k+1}}{q_{2k}} + \cfrac{\bar{d}_{2k+1}}{\bar{d}_{2k}}} < \frac{q_{2k}}{q_{2k+1}} < \frac{1}{k^3}. \tag{3. 11}$$

Again (1. 10) gives

$$q_{2k+1}\bar{d}_{2k+1} = \cfrac{1}{\cfrac{q_{2k+2}}{q_{2k+1}} + \cfrac{\bar{d}_{2k+2}}{\bar{d}_{2k+1}}}. \tag{3. 12}$$

(3. 10) gives at once

$$\frac{\bar{d}_{2k+2}}{\bar{d}_{2k+1}} < \frac{1}{k^3}$$

and from (3. 7) and (3. 8)

$$\frac{q_{2k+2}}{q_{2k+1}} = \frac{q_{2k+1} + q_{2k}}{q_{2k-1}} < 1 + \frac{1}{k^3};$$

putting this into (3. 12)

$$q_{2k+1}\bar{d}_{2k+1} > \frac{1}{1+\frac{2}{k^3}} > 1 - \frac{2}{k^3}. \tag{3. 13}$$

In order to extend the estimations (3. 11) and (3. 13) to all $s_\mu \bar{\delta}_\mu$'s we remark first that owing to (3. 4) all s''_μ's are at the same time q's, i. e. also with some k

$$s''_\mu \bar{\delta}''_\mu = q_{2k}\bar{d}_{2k} < \frac{1}{k^3}. \tag{3. 14}$$

As to the s'_μ's (3. 13) and Lemma III give for all s'_μ's with

$$q_{2k} < s'_\mu < q_{2k+1} \tag{3. 15}$$

the estimation

$$s'_\mu \bar{\delta}'_\mu > q_{2k+1}\bar{d}_{2k+1} > 1 - \frac{2}{k^3},$$

i. e.

$$s'_\mu \bar{\delta}'_\mu - 1 > -\frac{2}{k^3}. \tag{3. 16}$$

Now (3. 14) and (3. 3) give at once

$$\Sigma'_2 > -\frac{3}{2}\sum_{k=1}^{\infty}\frac{1}{k^3} > -3. \tag{3. 17}$$

To obtain a lower bound for Σ'_1 by the aid of (3. 16) we have to consider how many terms belong to the same k for any fixed k. The number of the s_ν-"Neben-Nenner" satisfying (3. 15) is owing to $a_{2k} = k^3$ obviously k^3; we need only an upper bound for the number of those which beside fulfilling (3. 15) also occur in the representation (1. 13) of N. Let these s'_μ's be

$$s_{\mu_1,k},\, s_{\mu_2,k},\, \ldots,\, s_{\mu_r,k}$$

where

$$s_{\mu_i,k} = q_{2k-1} + \mu_i q_{2k} \qquad (\mu_1 < \mu_2 < \cdots < \mu_r). \tag{3. 18}$$

We have to find an upper bound for r. Owing to the representation (1. 13) we have

$$s_{\mu_1,k} + \cdots + s_{\mu_{r-1},k} < s_{\mu_r,k} \leqq q_{2k+1} = q_{2k-1} + k^3 q_{2k}.$$

Using (3. 18) this gives a fortiori

$$\mu_1+\mu_2+\cdots+\mu_{r-1}<k^3.$$

Since

$$\mu_1+\mu_2+\cdots+\mu_{r-1}\geqq 1+2+\cdots+(r-1)=\frac{r(r-1)}{2},$$

we get

$$k^3>\frac{r(r-1)}{2}, \qquad r<3k^{3/2}.$$

Hence, by (3. 3) and (3. 16), we obtain

$$\Sigma_1'>-\sum_{k=1}^{\infty}3k^{3/2}\frac{1}{k^3}.$$

This and (3. 17) complete the proof.

Appendix

As told in the introduction, we shall prove here the first three lemmas.

Proof of Lemma I. We denote by Δ_ν the arc with the endpoints O and the $s_\nu\alpha$-point; then owing to the definition of l, Δ_ν and $\Delta_{\nu-1}$ have no common point except O. Let the $m\alpha$-point fall into $\Delta_\nu+\Delta_{\nu-1}$, then we have

$$m>s_\nu.$$

Since the length of the arc $\Delta_\nu+\Delta_{\nu-1}$ is $\delta_\nu+\delta_{\nu-1}$, there are two possibilities.

Case I. The length of the arc with the endpoints $s_\nu\alpha$-point and $m\alpha$-point (within $\Delta_\nu+\Delta_{\nu-1}$) is $\leqq\delta_{\nu-1}$.

Case II. The length of the arc with the endpoints $s_{\nu-1}\alpha$-point and $m\alpha$-point (within $\Delta_\nu+\Delta_{\nu-1}$) is $<\delta_\nu$.

In Case I the directed distance from the $s_\nu\alpha$-point to the $m\alpha$-point on the circle is the same as that from O to the $(m-s_\nu)\alpha$-point and this last point lies on $\Delta_{\nu-1}$ or in the $s_{\nu-1}\alpha$-endpoint. Owing to the definition of $s_{\nu-1}$ we have in this case $m-s_\nu\geqq s_{\nu-1}$, i. e.

(4. 1) $$m\geqq s_\nu+s_{\nu-1}.$$

In Case II the analogous reasoning gives

(4. 2) $$m>s_\nu+s_{\nu-1}.$$

The smallest m for which the $m\alpha$-point falls into $\Delta_\nu+\Delta_{\nu-1}$ is, according to the definition, $s_{\nu+1}$; from (4. 1) and (4. 2) it follows that

$$s_{\nu+1}\geqq s_\nu+s_{\nu-1}.$$

On the other hand, the $(s_\nu+s_{\nu-1})\alpha$-point lies on the arc $\Delta_\nu+\Delta_{\nu-1}$, indeed,

since the directed arc length on the circle from the $s_\nu\alpha$-point to the $(s_\nu+s_{\nu-1})\alpha$-point is the same as that from O to the $s_{\nu-1}\alpha$-point. This proves (1.4) and (1.5) consequently.

PROOF OF LEMMA II. It follows from the definition of the q_k's that the arc of the circle E which is bordered by O and the $q_{k-1}\alpha$-point, contains none of the α-, 2α-, ..., $q_k\alpha$-points. Hence, according to Lemma I,

$$s_{\nu_k+1}=q_{k-1}+q_k,\qquad \delta_{\nu_k+1}=d_{k-1}+d_k. \tag{4.3}$$

The remark after Lemma I in § 1 and the definition (1.3) of the digits a_k give that on the one hand the $s_{\nu_k+1}\alpha$-, ..., $s_{\nu_k+a_k}\alpha$-points lie on the same side of O as the $q_{k-1}\alpha$-point, and on the other hand the $s_{\nu_k+a_k+1}\alpha$-point on the opposite side, i. e.

$$q_{k+1}=s_{\nu_k+a_k}. \tag{4.4}$$

(4.3) and the repeated use of Lemma I give already (1.3) and, as easy to see, also (1.9). Owing to (4.4) the special case $r=a_k$ gives already (1.6) and (1.7).

Since $q_0=0$, $q_1=1$, $d_0=1$, $d_1=\alpha$ and from (1.6) and (1.7) $q_2=a_1$, $\bar{d}_2=1-a_1\alpha$, we have

$$q_1\bar{d}_2+q_2\bar{d}_1=1$$

and (1.10) follows from (1.6) and (1.7) by an easy induction.

PROOF OF LEMMA III. (1.11) follows clearly from the definition of the s_ν's, since if the $n\alpha$-point is not adjacent to O, this gives the existence of an integer $1\leq s_\nu<n$ for which the $s_\nu\alpha$-point is nearer to O than the $n\alpha$-point.

(1.12) follows from (1.8) and (1.9) in the following way:

$$s_\nu\bar{\delta}_\nu\equiv s_{\nu_k+r}\,\bar{\delta}_{\nu_k+r}=(q_{k+1}-(a_k-r)q_k)(\bar{d}_{k+1}+(a_k-r)\bar{d}_k)=$$
$$=q_{k+1}\bar{d}_{k+1}\left(1-(a_k-r)\frac{q_k}{q_{k+1}}\right)\left(1+(a_k-r)\frac{\bar{d}_k}{\bar{d}_{k+1}}\right).$$

On account of $\bar{d}_k>\bar{d}_{k+1}$, $q_k<\dfrac{1}{a_k}q_{k+1}$ and $0<a_k-r<a_k$

$$s_\nu\bar{\delta}_\nu>q_{k+1}\bar{d}_{k+1},$$

indeed.

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ЗАМЕЧАНИЕ О МЕХАНИЧЕСКОЙ КВАДРАТУРЕ

О. КИШ (Будапешт)

(Представлено П. Тураном)

П. Туран нашел в 1950-ом году следующую теорему:* Если r любое неотрицательное целое число, а $g(x)$ рациональный многочлен не более чем $2(r+1)n-1$-ой степени, то с соответствующими, независящими от $g(x)$ коэффициентами $\lambda_\nu^{(k)}$ имеет место квадратурная формула

$$\int_{-1}^{1}\frac{g(x)}{\sqrt{1-x^2}}dx=\sum_{\nu=0}^{n}\sum_{k=0}^{2r}\lambda_\nu^{(k)}\,g^{(k)}\left(\cos\frac{2\nu-1}{2n}\pi\right). \tag{1}$$

Отсюда следует, что, если $f(t)$ есть четный тригонометрический многочлен не более чем $2(r+1)n-1$-ого порядка, то имеет место квадратурная формула вида

$$\int_{0}^{\pi}f(t)\,dt=\sum_{\nu=0}^{n}\sum_{k=0}^{2r}\mu_\nu^{(k)}\,f^{(k)}\left(\frac{2\nu-1}{2n}\pi\right). \tag{2}$$

П. Туран поставил следующий вопрос: нельзя ли представить фигурирующие здесь коэффициенты $\mu_\nu^{(k)}$ в простой замкнутой форме? Нижеследующая теорема дает положительный ответ на этот вопрос.

Обозначим через $s_{r,\nu}$ $(\nu=0, 1, \ldots, r)$ элементарные симметрические многочлены чисел $1, 4, \ldots, r^2$, то-есть пусть

$$s_r=1,\ s_{r-1}=1+4+\cdots+r^2,\ s_{r-2}=1\cdot 4+1\cdot 9+\cdots+\\ +(r-1)^2r^2,\ \ldots,\ s_0=1\cdot 4\cdot\,\cdots\,\cdot r^2.$$

Имеет место следующая

Теорема. Если $f(t)$ есть четный тригонометрический многочлен не выше чем $2(r+1)n-1$-ого порядка, то

$$\int_{0}^{\pi}f(t)\,dt=\frac{\pi}{n\cdot r!^2}\sum_{\varrho=0}^{r}\frac{s_\varrho}{4^\varrho n^{2\varrho}}\sum_{\nu=1}^{n}f^{(2\varrho)}\left(\frac{2\nu-1}{2n}\pi\right). \tag{3}$$

Если $r=0$, то получается тот хорошо известный факт, согласно которому, если $f(t)$ есть четный тригонометрический многочлен не выше

* P. Turán, On the theory of the mechanical quadrature, *Acta Sci. Math. Szeged*, **12** (1950), стр. 30—37.

чем $2n-1$-ого порядка, то

$$\int_0^{\pi} f(t)\,dt = \frac{\pi}{n}\sum_{\nu=1}^{n} f\left(\frac{2\nu-1}{2n}\pi\right).$$

В случае $r=1$ речь идет о том, что для всякого четного тригонометрического многочлена не более чем $4n-1$-ого порядка имеет место равенство

$$\int_0^{\pi} f(t)\,dt = \frac{\pi}{n}\left\{\sum_{\nu=1}^{n} f\left(\frac{2\nu-1}{2n}\pi\right)+\frac{1}{4n^2}\sum_{\nu=1}^{n} f''\left(\frac{2\nu-1}{2n}\pi\right)\right\}.$$

Из теоремы можно сделать вывод и в связи с формулой (1). Если $g(x)$ есть рациональный многочлен не выше чем $2(r+1)n-1$-ой степени, то $g(\cos t)$ есть четный тригонометрический многочлен не более чем $2(r+1)n-1$-ого порядка,

$$\int_{-1}^{1}\frac{g(x)}{\sqrt{1-x^2}}\,dx=\int_0^{\pi} g(\cos t)\,dt$$

и поэтому, согласно теореме,

$$\int_{-1}^{1}\frac{g(x)}{\sqrt{1-x^2}}\,dx=\frac{\pi}{n\cdot r!^2}\sum_{\varrho=0}^{r}\frac{s_\varrho}{4^\varrho n^{2\varrho}}\sum_{\nu=1}^{n}\left[\frac{d^{2\varrho}}{dt^{2\varrho}}\,g(\cos t)\right]_{t=\frac{2\nu-1}{2n}\pi}.$$

Отсюда, произведя дифференцирование и перегруппировав члены суммы, получим формулу вида (1).

Так в случае $r=0$ для всех многочленов не выше $2n-1$-ой степени получаем хорошо известную формулу Эрмита:

$$\int_{-1}^{1}\frac{g(x)}{\sqrt{1-x^2}}\,dx=\frac{\pi}{n}\sum_{\nu=1}^{n} g\left(\cos\frac{2\nu-1}{2n}\pi\right),$$

в случае $r=1$ для многочленов не выше $4n-1$-ой степени имеем равенство

$$\int_{-1}^{1}\frac{g(x)}{\sqrt{1-x^2}}\,dx=\frac{\pi}{n}\left\{\sum_{\nu=1}^{n} g\left(\cos\frac{2\nu-1}{2n}\pi\right)-\right.$$

$$-\frac{1}{4n^2}\sum_{\nu=1}^{n}\cos\frac{2\nu-1}{2n}\pi\, g'\left(\cos\frac{2\nu-1}{2n}\pi\right)+$$

$$\left.+\frac{1}{4n^2}\sum_{\nu=1}^{n}\sin^2\frac{2\nu-1}{2n}\pi\, g''\left(\cos\frac{2\nu-1}{2n}\pi\right)\right\}.$$

Теорему можно сформулировать и в таком виде: если

$$f(t)=\sum_{k=0}^{2(r+1)n-1} a_k \cos kt,$$

то

$$a_0=\frac{1}{n\cdot r!^2}\sum_{\varrho=0}^{r}\frac{s_\varrho}{4^\varrho n^{2\varrho}}\sum_{\nu=1}^{n} f^{(2\varrho)}\left(\frac{2\nu-1}{2n}\pi\right),$$

то-есть a_0 может быть представлено с помощью значений функции и ее первых $2r$ производных в точках $\frac{2\nu-1}{2n}\pi$. Но, применяя теорему к функции $f(t)\cos kt$, сразу получаем, что, если

$$f(t)=\sum_{k=0}^{(r+1)n-1} a_k \cos kt,$$

то

$$a_k=\frac{2}{n\cdot r!^2}\sum_{\varrho=0}^{r}\frac{s_\varrho}{4^\varrho n^{2\varrho}}\sum_{\nu=1}^{n}\left[f(t)\cos kt\right]^{(2\varrho)}_{t=\frac{2\nu-1}{2n}\pi} \qquad (k=1, 2, \ldots, (r+1)n-1). \tag{4}$$

Теорема, аналогичная вышеприведенной, имеет место и в случае общих тригонометрических многочленов: если $f(t)$ есть тригонометрический многочлен не выше чем $(r+1)n-1$-ой степени, то

$$\int_0^{2\pi} f(t)\,dt=\frac{2\pi}{n\cdot r!^2}\sum_{\varrho=0}^{r}\frac{s_\varrho}{n^{2\varrho}}\sum_{\nu=0}^{n-1} f^{(2\varrho)}\left(\frac{2\pi\nu}{n}\right).$$

И в этом случае можно было бы написать для коэффициентов формулы, аналогичные соотношениям (4).

К случаю нечетных тригонометрических многочленов и к другим, примыкающим к сказанному, вопросам мы вернемся в другой заметке.

Переходя к доказательству теоремы, обозначим через $s[f(t)]$ правую часть равенства (3).

Так как встречающиеся операции линейны, достаточно доказать теорему для функций $f(t)=\cos pt$ $(p=0, 1, \ldots, 2(r+1)n-1)$, то-есть в виду того, что

$$\int_0^{\pi} dt=\pi, \quad s[1]=\pi, \quad \int_0^{\pi}\cos pt\,dt=0 \qquad (p=1, 2, \ldots)$$

достаточно доказать, что $s[\cos pt]=0$ $(p=1, 2, \ldots, 2(r+1)n-1)$.

Эти равенства имеют место, так как

$$s\cdot[\cos pt]=\frac{\pi}{n\cdot r!^2}\sum_{\varrho=0}^{r}(-1)^\varrho\,\frac{s_\varrho}{4^\varrho n^{2\varrho}}\,p^{2\varrho}\sum_{\nu=1}^{n}\cos p\,\frac{2\nu-1}{2n}\pi$$

и в случае $p=2n, 4n, \ldots, 2rn$ первая, а в случае $2n \nmid p$ вторая сумма равна нулю.

Два последних утверждения могут быть доказаны следующим образом. Пусть $p=2nq$ $(q=1,2,\ldots,r)$.

$$\sum_{\varrho=0}^{r}(-1)^{\varrho}\frac{s_\varrho}{4^\varrho n^{2\varrho}}p^{2\varrho}=\sum_{\varrho=0}^{r}(-1)^\varrho s_\varrho q^{2\varrho}=\prod_{\varrho=1}^{r}(\varrho^2-q^2)=0,$$

а это и есть наше первое утверждение.

Теперь мы докажем второе, что и завершит докавательство теоремы. Пусть $2n\nmid p$. Тогда

$$\sum_{\nu=1}^{n}\exp ip\frac{2\nu-1}{2n}\pi=\exp ip\frac{\pi}{2n}\cdot\frac{1-\exp ip\pi}{1-\exp\frac{ip\pi}{n}}=\begin{cases}0, & \text{если } p \text{ четно,}\\ \dfrac{2\exp\frac{ip\pi}{2n}}{1-\exp\frac{ip\pi}{n}}, & \text{если } p \text{ нечетно.}\end{cases}$$

Аналогичным образом можно показать, что

$$\sum_{\nu=1}^{n}\exp\left(-ip\frac{2\nu-1}{2n}\pi\right)=\begin{cases}0, & \text{если } p \text{ четно,}\\ \dfrac{2\exp\left(-\frac{ip\pi}{2n}\right)}{1-\exp\left(-\frac{ip\pi}{n}\right)}=-\dfrac{2\exp\frac{ip\pi}{2n}}{1-\exp\frac{ip\pi}{n}}, & \text{если } p \text{ нечетно.}\end{cases}$$

Поэтому, действительно,

$$\sum_{\nu=1}^{n}\cos p\frac{2\nu-1}{2n}\pi=\frac{1}{2}\left\{\sum_{\nu=1}^{n}\exp ip\frac{2\nu-1}{2n}\pi+\sum_{\nu=1}^{n}\exp\left(-ip\frac{2\nu-1}{2n}\pi\right)\right\}=0.$$

В заключение выражаю свою благодарность академику П. Турану за ценную помощь.

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REPRESENTATIONS FOR REAL NUMBERS AND THEIR ERGODIC PROPERTIES

By

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Introduction

We shall consider representations of a real number x by infinite iteration of a positive function $y=f(x)$ in the form of the "f-expansion"

(1) $$x=\varepsilon_0+f(\varepsilon_1+f(\varepsilon_2+f(\varepsilon_3+\cdots)\ldots)$$

where the "digits" $\varepsilon_n=\varepsilon_n(x)$ $(n=0, 1, \ldots)$ and the "remainders"

(2) $$r_n(x)=f(\varepsilon_{n+1}+f(\varepsilon_{n+2}+f(\varepsilon_{n+3}+\cdots)\ldots) \qquad (n=0, 1, \ldots)$$

are defined by the following recursive relations:

(3) $$\begin{aligned} \varepsilon_0(x)&=[x], & r_0(x)&=(x), \\ \varepsilon_{n+1}(x)&=[\varphi(r_n(x))], & r_{n+1}(x)&=(\varphi(r_n(x))) \qquad (n=0, 1, \ldots) \end{aligned}$$

where $[z]$ denotes the integral part and (z) the fractional part of the real number z and $x=\varphi(y)$ is the inverse function of $y=f(x)$. In § 1 we shall investigate what conditions imposed on the function $f(x)$ are sufficient to ensure that every real number x should have a representation in the form of the f-expansion (1).[1]

The representation (1) reduces for $f(x)=\frac{x}{q}$ $(q=2, 3, \ldots)$ to the q-adic expansion $x=\sum_{n=0}^{\infty}\frac{\varepsilon_n}{q^n}$ and for $f(x)=\frac{1}{x}$ to the continued fraction representation of x. The case when $f(x)$ is a general decreasing function has been considered previously by B. H. BISSINGER [1]. Our treatment is still more general than his, since we do not suppose the unnecessary condition that $f(x)$ is positive for any $x\geqq 1$ (i. e. that $\varphi(0)=+\infty$). The case when $f(x)$ is a general increasing function has been considered previously by C. I. EVERETT [2]. He supposed the unnecessary condition that $\varphi(1)$ is an integer. We shall not need this restriction. The principal aim of the present paper, however, is not this generalization of the conditions ensuring the validity of

[1] If for some n we have $r_n(x)=0$, then $r_{n+k}(x)$ and $\varepsilon_{n+k}(x)$ are not defined for $k=1, 2, \ldots$, and x has the finite representation $x=\varepsilon_0+f(\varepsilon_1+f(\varepsilon_2+\cdots+f(\varepsilon_n)\ldots)$.

the representation (1), but to prove some theorems on the ergodic properties of the digits $\varepsilon_n(x)$ and the remainders $r_n(x)$ which contain as special cases the well-known theorems on q-adic expansions and on continued fractions, respectively (see [5]—[15]). To obtain such theorems we have to impose some additional restrictions on $f(x)$.

The mentioned ergodic properties of an "f-expansion (1) with independent digits" will be investigated in § 2. In § 3 we consider some examples in which our general theorem is applicable; q-adic expansions, continued fractions and the algorithm of W. Bolyai (see [2], [3], [4]). In § 4 we consider a class of f-expansions, called β-adic expansions ($\beta > 1$ not an integer), to which our theorem can not be applied, but another method leads to the same conclusion.[2]

§ 1. Representation theorems

A) We consider first the case when $f(x)$ is a decreasing function. We suppose

A1) $f(1)=1$.

We suppose further

A2) *$f(t)$ is positive, continuous and strictly decreasing for* $1 \leq t \leq T$ *and* $f(t)=0$ *for* $t \geq T$ *where* $2 < T \leq +\infty$ (in case $T=+\infty$, this means that $\lim\limits_{t\to+\infty} f(t)=0$).

We distinguish three subcases:

A2$_1$) $T=+\infty$; A2$_2$) $2<T<+\infty$ and T is an integer; A2$_3$) $2<T<+\infty$ and T is not an integer.

Let us mention that B. H. Bissinger considered only the case A2$_1$).

Following Bissinger, we suppose further that the following condition is also satisfied:[3]

A3) $|f(t_2)-f(t_1)| < |t_2-t_1|$ *for* $1 \leq t_1 < t_2$ *and there is a constant* λ *such that* $0<\lambda<1$ *and*

$$|f(t_2)-f(t_1)| \leq \lambda |t_2-t_1| \quad \textit{if} \quad 1+f(2)<t_1<t_2.$$

We shall prove that conditions A1), A2) and A3) imply that the representation (1) is valid for any real x. (Clearly, it suffices to prove this for $0<x<1$. In what follows we shall always suppose therefore that $0<x<1$.)

[2] The assertions of Theorem 1 have been proved under somewhat more restrictive suppositions and Theorem 2 has been announced without proof in a previous paper (in Hungarian language) [16] of the author.

[3] This condition could be replaced by a less restrictive one as will be pointed out below.

Before proving this, we introduce some notations. Let us define

$$f_1(z_1) = f(z_1),$$
$$(1.1)\qquad f_n(z_1, z_2, \ldots, z_n) = f_{n-1}(z_1, z_2, \ldots, z_{n-2}, z_{n-1} + f(z_n))$$

for $n = 2, 3, \ldots$. Let us put further

$$(1.2)\qquad C_n(x) = f_n(\varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_n(x))$$

where the digits $\varepsilon_1(x), \varepsilon_2(x), \ldots$ are defined by the recursion (3). We shall call $C_n(x)$ the n-th convergent of x. The validity of (1) means that either we have $r_n(x) = 0$ for some n, in which case $x = f(\varepsilon_1 + f(\varepsilon_2 + \cdots + f(\varepsilon_n)\ldots))$, or

$$(1.3)\qquad \lim_{n\to\infty} C_n(x) = x.$$

We have to consider only the latter case when $r_n(x) \neq 0$ $(n = 1, 2, \ldots)$. We have clearly (for $0 < x < 1$)

$$(1.4)\qquad x = f_n(\varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_{n-1}(x), \varepsilon_n(x) + r_n(x)).$$

Thus it follows

$$(1.5)\qquad x - C_n(x) = f_n(\varepsilon_1(x), \ldots, \varepsilon_n(x) + r_n(x)) - f_n(\varepsilon_1(x), \ldots, \varepsilon_n(x)),$$

and therefore putting

$$(1.6)\qquad u_k = \varepsilon_{k+1}(x) + f_{n-k-1}(\varepsilon_{k+2}(x), \ldots, \varepsilon_n(x) + r_n(x))$$

and

$$v_k = \varepsilon_{k+1}(x) + f_{n-k-1}(\varepsilon_{k+2}(x), \ldots, \varepsilon_n(x))$$

for $k = 0, 1, \ldots, n-1$, we have

$$(1.7)\qquad x - C_n(x) = r_n(x) \prod_{k=0}^{n-1} \left(\frac{f(u_k) - f(v_k)}{u_k - v_k} \right).$$

Now each factor on the right of (1.7) has an absolute value not exceeding 1. We shall prove that from any two numbers

$$\left| \frac{f(u_k) - f(v_k)}{u_k - v_k} \right|, \quad \left| \frac{f(u_{k+1}) - f(v_{k+1})}{u_{k+1} - v_{k+1}} \right| \qquad (k = 0, 1, \ldots, n-3)$$

at least one does not exceed λ. As a matter of fact, we have

$$u_k = \varepsilon_{k+1} + f(\varepsilon_{k+2} + f(u_{k+2})),$$
$$u_{k+1} = \varepsilon_{k+2} + f(u_{k+2}),$$

and similarly

$$v_k = \varepsilon_{k+1} + f(\varepsilon_{k+2} + f(v_{k+2})),$$
$$v_{k+1} = \varepsilon_{k+2} + f(v_{k+2}).$$

Three cases are possible. If $\varepsilon_{k+1} \geqq 2$, then $u_k \geqq 2 > 1 + f(2)$ and $v_k \geqq 2 > 1 + f(2)$ and thus by condition A3) $\left| \frac{f(u_k) - f(v_k)}{u_k - v_k} \right| \leqq \lambda$. If $\varepsilon_{k+1} = 1$

and $\varepsilon_{k+2} \geqq 2$, then similarly we obtain $\left|\dfrac{f(u_{k+1})-f(v_{k+1})}{u_{k+1}-v_{k+1}}\right| \leqq \lambda$. Finally, if $\varepsilon_{k+1} = \varepsilon_{k+2} = 1$, then

$$u_k = 1+f(1+f(u_{k+2})) \geqq 1+f(2)$$

and

$$v_k = 1+f(1+f(v_{k+2})) \geqq 1+f(2).$$

Thus our assertion is proved. It follows from (1.7) that

$$|x-C_n(x)| \leqq \lambda^{\left[\frac{n}{2}\right]-1} \tag{1.8}$$

and (1.8) clearly implies (1.3).

The above proof is essentially that of BISSINGER. By the same method it can be shown that it suffices to suppose that $\left|\dfrac{f(t_2)-f(t_1)}{t_2-t_1}\right| \leqq \lambda < 1$ holds for $t_2 > t_1 \geqq 1+f_{2r-1}(1, 1, \ldots, 1, 2)$ for some r $(r = 1, 2, 3, \ldots)$, because in this case from $2r$ consecutive numbers $\left|\dfrac{f(u_k)-f(v_k)}{u_k-v_k}\right|$ at least one does not exceed λ.

B) Now we consider the case when $f(x)$ is increasing. We suppose first of all

B1) $f(0) = 0$.

We suppose further that the following condition is satisfied:

B2) $f(t)$ *is continuous and strictly increasing for* $0 \leqq t \leqq T$ *and* $f(t) = 1$ *if* $t \geqq T$ *where* $1 < T \leqq +\infty$. (In case $T = +\infty$, this means $\lim_{t\to+\infty} f(t) = 1$.)

We distinguish again three subcases: $B2_1$), $B2_2$), $B2_3$) accordingly as $T = +\infty$, $T < +\infty$ and T is an integer, $T < +\infty$ and T is not an integer, respectively. EVERETT considered only the case $B2_2$).

We need here also a condition on the slope $\dfrac{f(t_2)-f(t_1)}{t_2-t_1}$. For example, the following condition considered already by EVERETT is *sufficient:*[4]

B3) $\dfrac{f(t_2)-f(t_1)}{t_2-t_1} < 1$ *for* $0 \leqq t_1 < t_2$.

If B1), B2) and B3) are satisfied, then the f-expansion (1) is valid for any real x. (We may suppose again $0 \leqq x < 1$.) Following EVERETT, this can be shown as follows:

Clearly, the sequence $C_n(x)$ $(n = 1, 2, \ldots)$ defined by (1.2) is non-decreasing and the sequence $D_n(x)$, where $D_n(x)$ is defined as the least value of $C_n(x')$ which is greater than $C_n(x)$ (or 1 if such an x' does not exist), is

[4] This condition can be replaced by a weaker one, cf. [2].

non-increasing and

(1.9) $$C_n(x) \leqq x < D_n(x).$$

Thus

(1.10) $$\underline{x} = \lim_{n\to\infty} C_n(x)$$

and

(1.11) $$\bar{x} = \lim_{n\to\infty} D_n(x)$$

always exist and $\underline{x} \leqq x \leqq \bar{x}$. We have to prove that $\underline{x} = \bar{x} = x$ for any x $(0<x<1)$. If this would not hold for all x in $(0, 1)$, then there would exist a finite or denumerable sequence of non-overlapping „gaps" $(\underline{x}, \bar{x})$ in the unit interval, and thus there would exist an x for which $\bar{x}-\underline{x}$ is maximal. For this value of x we would have by condition B3) putting $r_1(x)=y$

(1.12) $$\bar{x}-\underline{x} = \left(\frac{f(\varepsilon_1(x)+\bar{y})-f(\varepsilon_1(x)+\underline{y})}{\bar{y}-\underline{y}}\right)(\bar{y}-\underline{y}) < \bar{y}-\underline{y}$$

which contradicts our assumption that $\bar{x}-\underline{x}$ is maximal. Thus we have $\underline{x} = x = \bar{x}$ for all x.[5]

The admissible values for $\varepsilon_n(x)$ $(n=1, 2, \ldots)$ are $1, 2, \ldots$ in case A2_1), $1, 2, \ldots, T-1$ in case A2_2) and $1, 2, \ldots, [T]$ in case A2_3), similarly $0, 1, \ldots$ in case B2_1), further $0, 1, \ldots, T-1$ in case B2_2) and $0, 1, \ldots, [T]$ in case B2_3). Let us call a finite sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ a *canonical sequence* with respect to a given function $f(x)$, which satisfies either conditions A1), A2) and A3) or conditions B1), B2) and B3), if there exists a number x $(0 \leqq x < 1)$ such that $\varepsilon_k(x) = \varepsilon_k$ $(k=1, 2, \ldots, n)$. There is an essential difference for decreasing $f(x)$ between the case when T is an integer or $T=+\infty$ (cases A2_1) and A2_2)) and, on the other hand, the case with a finite non-integral T (case A2_3)). This difference consists in that in the case of an integer T or $T=+\infty$ all finite sequences $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ consisting of admissible digits, i. e. all sequences of positive integers $<T$ are canonical, while in the case when T is not an integer this is not true. The same difference

[5] J. Czipszer remarked that the above method of the proof, due to Everett, may be combined with the method of Bissinger in the case when $f(x)$ is decreasing, and in this way it can be shown that condition A3) can be replaced by the following weaker condition:

A3*) $|f(t_2)-f(t_1)| \leqq |t_2-t_1|$ for $1 \leqq t_1 < t_2$

and

$$|f(t_2)-f(t_1)| < |t_2-t_1| \quad \text{if} \quad \tau-\varepsilon < t_1 < t_2$$

where τ is the solution of the equation $1+f(\tau)=\tau$ and $0<\varepsilon<\tau$ is arbitrary. The only essential difference in the proof consists in that $\underline{x}$ and $\bar{x}$ are defined as $\underline{x} = \lim_{n\to\infty} C_{2n}(x)$ and $\bar{x} = \lim_{n\to\infty} C_{2n+1}(x)$, respectively.

exists for increasing $f(x)$ between the case when T is an integer or $T=+\infty$ (cases B2$_1$) and B2$_2$)) and the case when T is finite but not an integer (case B2$_3$)). While in cases B2$_1$) and B2$_2$) every finite sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ of non-negative integers $< T$ is canonical,[6] this is not true in case B2$_3$). By other words, in both cases A) and B) if T is an integer or $T=+\infty$, the values of the digits ε_n of a canonical sequence can be chosen independently, but if T is finite and not an integer, there exists some dependence between the members of a canonical sequence.

We shall call the f-expansions when one of the conditions A2$_1$), A2$_2$) respectively B2$_1$), B2$_2$) is satisfied *f-expansions with independent digits*, and the f-expansions when A2$_3$) respectively B2$_3$) are satisfied *f-expansions with dependent digits*. It should be noted that independence is not meant here in the sense of probability theory, but only in a weaker sense. As a matter of fact, in some cases, (e. g., in the case of the q-adic expansions) the digits $\varepsilon_n(x)$ considered as random variables (on the interval (0, 1) with the Lebesgue measure) are also statistically independent but for most f-expansions with independent digits this is not true. (For example, the digits of a continued fraction are not statistically independent.)

We shall see that the investigation of ergodic properties of f-expansions is much easier for f-expansions with independent digits than for f-expansions with dependent digits. The first case will be considered in § 2; in § 3 the ergodic theory of some special f-expansions with dependent digits, called the β-expansions, and corresponding to $f(x)=\frac{x}{\beta}$ for $0\leq x\leq\beta$ ($\beta>1$ non-integral) is investigated.

§ 2. Ergodic theory of f-expansions with independent digits

In this § we consider only f-expansions with independent digits. Let $f(x)$ satisfy the corresponding conditions of § 1. Then $f(x)$ is derivable almost everywhere and absolutely continuous. Clearly the same holds for $f_n(\varepsilon_1, \ldots, \varepsilon_n+t)$ as a function of t $(0\leq t\leq 1)$.

Let us put

$$(2.1)\qquad H_n(x,t)=\frac{d}{dt}f_n(\varepsilon_1(x), \ldots, \varepsilon_{n-1}(x), \varepsilon_n(x)+t).$$

Then $H_n(x, t)$ is defined for any x, for which $\varepsilon_n(x)$ is defined,[7] and for almost

[6] In these cases clearly $D_n(x)=f_n(\varepsilon_1(x), \ldots, \varepsilon_{n-1}(x), \varepsilon_n(x)+1)$.

[7] I. e., except for those x which have a finite representation in the form (1) of length smaller than n.

all t. We shall suppose that $f(x)$ satisfies also the following condition:

$$\text{C)} \qquad \frac{\sup\limits_{0<t<1} |H_n(x, t)|}{\inf\limits_{0<t<1} |H_n(x, t)|} \leq C$$

where the constant $C \geq 1$ does not depend neither on x nor on n.

We prove the following

THEOREM 1. *If $f(x)$ satisfies the conditions* A1), A2$_1$) *or* A2$_2$), A3) *and* C); *or the conditions* B1), B2$_1$) *or* B2$_2$), B3) *and* C), *respectively, then for any function $g(x)$ which is L-integrable in the interval* (0, 1) *we have for almost all x*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(r_k(x)) = M(g), \tag{2.2}$$

where $M(g)$ is a finite constant which can be represented in the form

$$M(g) = \int_0^1 g(x)\, h(x)\, dx \tag{2.3}$$

where $h(x)$ is a measurable function, depending only on $f(x)$ and satisfying the inequality

$$\frac{1}{C} \leq h(x) \leq C \tag{2.4}$$

where C is the constant figuring in condition C). *The measure*

$$\nu(E) = \int_E h(x)\, dx \tag{2.5}$$

is invariant with respect to the transformation

$$Tx = (\varphi(x)) \qquad (0 < x < 1) \tag{2.6}$$

where $y = \varphi(x)$ is the inverse function of $x = f(y)$.

PROOF. Let $\mathfrak{E}_n = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ denote a canonical sequence of n terms with respect to $f(x)$. The intervals $(f_n(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n), f_n(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n + 1))$ do not overlap and if $\mathfrak{E}_n$ runs over all canonical sequences of n terms, these intervals fill out the interval (0, 1). Therefore we have

$$\sum_{\mathfrak{E}_n} |f_n(\varepsilon_1, \ldots, \varepsilon_{n-1}, \varepsilon_n + 1) - f_n(\varepsilon_1, \ldots, \varepsilon_{n-1}, \varepsilon_n)| = 1 \tag{2.7}$$

where the summation is to be extended over all canonical sequences $\mathfrak{E}_n$ of n terms.

Let us consider the mapping $Tx = (\varphi(x))$ of the interval (0, 1) onto itself. For any subset E of (0, 1) we denote by $T^{-1}E$ the set of those real numbers x $(0 < x < 1)$ for which $Tx \in E$. We define further $T^{-n}E$ by the recur-

sion: $T^{-n}E = T^{-1}(T^{-(n-1)}E)$ $(n=2, 3, \ldots)$. Clearly $T^{-n}E$ is measurable if E is any measurable subset of $(0, 1)$. Let $I_{a,b}$ denote the interval (a, b) $(0<a<b<1)$ and let $\mu(E)$ denote the Lebesgue measure of the set E. Then we have clearly

$$\mu(T^{-n}I_{a,b}) = \sum_{\mathfrak{E}_n} |f_n(\varepsilon_1, \ldots, \varepsilon_n + b) - f_n(\varepsilon_1, \ldots, \varepsilon_n + a)| \tag{2.8}$$

where the summation is to be extended again over all canonical sequences $\mathfrak{E}_n = (\varepsilon_1, \ldots, \varepsilon_n)$ of n terms. Let us denote by $x(\mathfrak{E}_n)$ a number for which

$$\varepsilon_k(x(\mathfrak{E}_n)) = \varepsilon_k \qquad (k=1, 2, \ldots, n); \tag{2.9}$$

such a number $x(\mathfrak{E}_n)$ exists for any canonical sequence $\mathfrak{E}_n$ by definition. It follows from (2. 7) that

$$\sum_{\mathfrak{E}_n} \inf_{0<t<1} |H_n(x(\mathfrak{E}_n), t)| \leq 1 \leq \sum_{\mathfrak{E}_n} \sup_{0<t<1} |H_n(x(\mathfrak{E}_n), t)| \tag{2.10}$$

and from (2. 8) that

$$\sum_{\mathfrak{E}_n} \inf_{0<t<1} |H_n(x(\mathfrak{E}_n), t)| \leq \frac{\mu(T^{-n}I_{a,b})}{(b-a)} \leq \sum_{\mathfrak{E}_n} \sup_{0<t<1} |H_n(x(\mathfrak{E}_n), t)|. \tag{2.11}$$

Comparing (2. 10) and (2. 11) we obtain by condition C) that

$$\frac{1}{C}\mu(E) \leq \mu(T^{-n}E) \leq C\mu(E), \tag{2.12}$$

provided that E is a subinterval of (0, 1). It follows easily that (2. 12) holds for any measurable subset E of the interval (0, 1). Thus we have

$$\frac{1}{C}\mu(E) \leq \frac{1}{n}\sum_{k=0}^{n-1}\mu(T^{-k}E) \leq C\mu(E) \qquad (n=1, 2, \ldots) \tag{2.13}$$

where $C \geq 1$ does not depend on n. According to the theorem of DUNFORD and MILLER ([17], [18]), it follows from the upper inequality of (2. 13) that for any L-integrable function $g(x)$ the limit

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} g(T^k x) = g^*(x) \tag{2.14}$$

exists for almost all x. But clearly $T^k x = r_k(x)$ $(k=0, 1, \ldots)$ and thus we obtain

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} g(r_k(x)) = g^*(x) \tag{2.15}$$

for almost all x.

To prove that $g^*(x)$ is (almost everywhere) equal to a constant depending only on $g(x)$, by a well-known argument it suffices to prove that the

transformation T is *ergodic* (indecomposable), or by other words, that if E is a measurable invariant set of positive measure, i. e. $T^{-1}E=E$ and $\mu(E)>0$, then $\mu(E)=1$

According to a theorem of K. KNOPP [19], if $\mu(E)>0$ and there exists a class J of subintervals of $(0, 1)$ such that a) every open subinterval of $(0, 1)$ is the union of a finite or a denumerably infinite sequence of disjoint intervals belonging to J and b) for any $I\in J$ we have $\mu(EI)\geq \Delta\mu(I)$ where $\Delta>0$ does not depend on I, then $\mu(E)=1$. We shall show that the class J of all intervals $I_{\mathcal{E}_n}=[f_n(\varepsilon_1,\ldots,\varepsilon_n), f_n(\varepsilon_1,\ldots,\varepsilon_n+1))=[a_{\mathcal{E}_n}, b_{\mathcal{E}_n})$ where $\mathcal{E}_n=(\varepsilon_1,\ldots,\varepsilon_n)$ is a canonical sequence $(n=1, 2,\ldots)$ has the properties required by the mentioned theorem of KNOPP. The class J has according to the representation theorems of § 1 the property a). As regards b), let us put

$$E(x)=\begin{cases} 1 & \text{if} \quad x\in E, \\ 0 & \text{if} \quad x\bar{\in} E. \end{cases} \tag{2. 16}$$

Then we have

$$\mu(EI_{\mathcal{E}_n})=\int\limits_{a_{\mathcal{E}_n}}^{b_{\mathcal{E}_n}} E(x)\,dx. \tag{2. 17}$$

Introducing in the integral on the right of (2. 17) the new variable t defined by $x=f_n(\varepsilon_1,\ldots,\varepsilon_n+t)$ (i. e. putting $t=r_n(x)=T^nx$) and taking into account that by virtue of the supposition $T^{-1}E=E$ we have $E(T^{-n}x)=E(x)$, further that $\dfrac{dx}{dt}=H_n(x(\mathcal{E}_n), t)$ where $x(\mathcal{E}_n)$ is a number for which $\varepsilon_k(x(\mathcal{E}_n))=\varepsilon_k$ $(k=1, 2,\ldots, n)$, we obtain

$$\mu(EI_{\mathcal{E}_n})=\int\limits_0^1 E(t)|H_n(x(\mathcal{E}_n), t)|\,dt. \tag{2. 18}$$

It follows by condition C) that

$$\mu(EI_{\mathcal{E}_n})\geqq\mu(E)\inf_{0<t<1}|H_n(x(\mathcal{E}_n), t)|\geqq\frac{\mu(E)}{C}\sup_{0<t<1}|H_n(x(\mathcal{E}_n), t)|. \tag{2. 19}$$

On the other hand,

$$\sup_{0<t<1}|H_n(x(\mathcal{E}_n), t)|\geqq\int\limits_0^1|H_n(x(\mathcal{E}_n), t)|\,dt=\mu(I_{\mathcal{E}_n}). \tag{2. 20}$$

Thus we obtain from (2. 19) and (2. 20)

$$\mu(EI_{\mathcal{E}_n})\geqq\frac{\mu(E)}{C}\mu(I_{\mathcal{E}_n}), \tag{2. 21}$$

i. e. the property b) of KNOPP's theorem holds for the class J. Thus T is ergodic, and therefore $g^*(x) = M(g)$ is constant almost everywhere. It remains to prove the existence of the function $h(x)$ satisfying (2. 3) and (2. 4), and the invariance of the measure $\nu(E) = \int_E h(x)\,dx$ with respect to the transformation T.

Let us put for any measurable subset E of $(0, 1)$

$$E(x) = \begin{cases} 1 & \text{for} \quad x \in E, \\ 0 & \text{for} \quad x \notin E \end{cases}$$

and

$$\nu_n(E) = \frac{1}{n} \sum_{k=0}^{n-1} \mu(T^{-k}E) = \int_0^1 \left(\frac{1}{n} \sum_{k=0}^{n-1} E(T^k x) \right) dx. \tag{2. 22}$$

As $0 \leq E(x) \leq 1$, it follows from the existence almost everywhere of the limit (2. 2) proved above for $g(x) = E(x)$ and LEBESGUE's theorem, that

$$\lim_{n \to \infty} \nu_n(E) = \nu(E) \tag{2. 23}$$

exists for any measurable E. As by (2. 13)

$$\frac{1}{C} \mu(E) \leq \nu(E) \leq C \mu(E), \tag{2. 24}$$

$\nu(E)$ is a measure which is equivalent to the Lebesgue measure $\mu(E)$; the ν-measure of the interval $(0, 1)$ is evidently equal to 1.

It follows by (2. 22)

$$\nu_n(T^{-1}E) = \frac{n+1}{n} \nu_{n+1}(E) - \frac{\mu(E)}{n} \tag{2. 25}$$

and therefore

$$\nu(T^{-1}E) = \nu(E), \tag{2. 26}$$

i. e. ν is invariant with respect to the transformation T.

Let us put

$$h(x) = \frac{dV(x)}{dx} \tag{2. 27}$$

where $V(x) = \nu(I_{0,x})$; here $I_{0,x}$ denotes the interval $(0, x)$ $(0 < x \leq 1)$.

From the invariance of the measure ν with respect to T it follows, as well known, that

$$M(g) = \int_0^1 g(x)\, h(x)\, dx. \tag{2. 28}$$

Thus (2. 3) is proved. (2. 4) follows evidently from (2. 24). Thus Theorem 1 is completely proved.

Let us define the function $e_k(x)$ as follows: If $f(x)$ is decreasing, put for $1 \leqq k < T$

$$e_k(x) = \begin{cases} 1 & \text{for} \quad f(k+1) < x \leqq f(k), \\ 0 & \text{otherwise.} \end{cases}$$

If $f(x)$ is increasing, put for $0 \leqq k < T$

$$e_k(x) = \begin{cases} 1 & \text{for} \quad f(k) \leqq x < f(k+1), \\ 0 & \text{otherwise.} \end{cases}$$

Applying our theorem to $g(x) = e_k(x)$ it follows that the relative frequency of every admissible digit converges to a positive limit, for almost all x, and these limits depend only on the function $f(x)$ and not on x. The values of these limits can be calculated for a given $f(x)$ if we succeed in constructing explicitly the corresponding (uniquely determined) invariant measure ν.

§ 3. Some examples

EXAMPLE 1. Let us put

$$f(x) = \begin{cases} \dfrac{x}{q} & \text{for} \quad 0 \leqq x \leqq q, \\ 1 & \text{for} \quad x > q \end{cases}$$

where $q \geqq 2$ is an integer. Clearly conditions B1), B2$_2$) and B3) are satisfied, further condition C) is also satisfied (with $C = 1$) because $H_n(x, t)$ is identically equal to $\dfrac{1}{q^n}$. Thus we obtain as a special case of our Theorem 1 the theorem of RAIKOFF [6] and the classical theorem of BOREL [5] on normal decimals, respectively. In this special case $\nu(E) = \mu(E)$, i. e. the Lebesgue measure is invariant with respect to the tranformation $Tx = (qx)$.

EXAMPLE 2. Let us put $f(x) = \dfrac{1}{x}$ for $x \geqq 1$. Clearly conditions A1), A2$_1$) and A3) are satisfied. To show that condition C) is also satisfied, we need the well-known formula according to which if $\dfrac{p_k(x)}{q_k(x)}$ denotes the k-th convergent of the continued fraction of x, we have

$$f_n(\varepsilon_1(x), \ldots, \varepsilon_n(x) + t) = \frac{p_{n-1}(x)(\varepsilon_n(x) + t) + p_{n-2}(x)}{q_{n-1}(x)(\varepsilon_n(x) + t) + q_{n-2}(x)}.$$

It follows that

$$H_n(x, t) = \frac{(-1)^n}{(q_{n-1}(x)(\varepsilon_n(x) + t) + q_{n-2}(x))^2}$$

and thus

$$\frac{\sup\limits_{0<t<1} |H_n(x,t)|}{\inf\limits_{0<t<1} |H_n(x,t)|} = \left(1+\frac{q_{n-1}(x)}{q_n(x)}\right)^2 \leqq 4.$$

Consequently, condition C) is satisfied with $C=4$ and therefore[8] by (2.12) $\mu(T^{-n}E) \leqq 4\mu(E)$. Thus we obtain as a special case of Theorem 1 the theorem of RYLL-NARDZEWSKI [12].

EXAMPLE 3. Let us consider the case when $f(x) = \sqrt[m]{1+x}-1$ for $0 \leqq x \leqq 2^m-1$ where $m \geqq 2$ is an integer. Conditions B1), B2) and B3) are clearly satisfied and thus every real number x can be represented in the form

$$x = \varepsilon_0 - 1 + \sqrt[m]{\varepsilon_1 + \sqrt[m]{\varepsilon_2 + \sqrt[m]{\varepsilon_3 + \cdots}}}$$

where the digits ε_n are generated by the recursion

$$\varepsilon_0 = [x], \qquad r_0 = (x),$$

$$\varepsilon_{n+1} = [(1+r_n)^m - 1], \quad r_{n+1} = ((1+r_n)^m - 1) \qquad (n = 0, 1, \ldots),$$

and thus the digits ε_n are capable of the values $0, 1, \ldots, 2^m-2$. This algorithm may be called the algorithm of W. BOLYAI who used it to approximate the roots of some equations (in the special case $m=2$) in his book "Tentamen..." [3] published in the year 1832.

Let us verify that condition C) is fulfilled. We have clearly

$$\frac{\sup\limits_{0<t<1} H_n(x,t)}{\inf\limits_{0<t<1} H_n(x,t)} = \prod_{j=1}^{n} \left(\frac{\varepsilon_j + \sqrt[m]{\varepsilon_{j+1} + \sqrt[m]{\varepsilon_{j+2} + \cdots + \sqrt[m]{\varepsilon_n + 2}}}}{\varepsilon_j + \sqrt[m]{\varepsilon_{j+1} + \sqrt[m]{\varepsilon_{j+2} + \cdots + \sqrt[m]{\varepsilon_n + 1}}}} \right)^{1-\frac{1}{m}}.$$

Thus, owing to the inequality $\dfrac{a+c}{a+b} \leqq \dfrac{c}{b}$ if $0 < b \leqq c$ and $a \geqq 0$, it follows

$$\frac{\sup\limits_{0<t<1} H_n(x,t)}{\inf\limits_{0<t<1} H_n(x,t)} \leqq \prod_{j=1}^{n} \left(1+\frac{1}{\varepsilon_n+1}\right)^{\left(1-\frac{1}{m}\right)\cdot\frac{1}{m^{n-j}}} \leqq 2,$$

i. e. condition C) is satisfied with $C=2$.

[8] It has been shown by HARTMAN that more is true; we have $\mu(T^{-n}E) \leqq 2\mu(E)$ (see [14] and for another proof [16]).

§ 4. The β-expansion of real numbers

In this § we consider the case

$$f(x)=\begin{cases}\dfrac{x}{\beta} & \text{for} \quad 0\leqq x\leqq\beta,\\ 1 & \text{for} \quad \beta<x\end{cases}$$

where $\beta>1$ is not an integer. As conditions B1), B2$_3$) and B3) are clearly satisfied, it follows that every real number x can be represented in the form

$$x=\varepsilon_0+\frac{\varepsilon_1}{\beta}+\frac{\varepsilon_2}{\beta^2}+\cdots+\frac{\varepsilon_n}{\beta^n}+\cdots \tag{4.1}$$

where the digits ε_n can be obtained by the recursion formulae

$$\begin{aligned}\varepsilon_0&=[x], \quad r_0=(x),\\ \varepsilon_{n+1}&=[\beta r_n], \quad r_{n+1}=(\beta r_n) \qquad (n=0,1,\ldots).\end{aligned} \tag{4.2}$$

The digits ε_n which for $n\geqq 1$ are capable of the values $0, 1, \ldots, [\beta]$ can be expressed without introducing the remainders r_n as

$$\begin{aligned}\varepsilon_0&=[x],\\ \varepsilon_1&=[\beta(x)],\\ \varepsilon_2&=[\beta(\beta(x))],\\ \varepsilon_3&=[\beta(\beta(\beta(x)))],\\ &\vdots\end{aligned} \tag{4.3}$$

In this case Tx is the transformation $Tx=(\beta x)$ of the interval $(0, 1)$ onto itself.

We shall prove

THEOREM 2. *For any function $g(x)$ which is L-integrable in $(0, 1)$ we have for almost all x*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}g(r_k(x))=M(g) \tag{4.4}$$

where the constant $M(g)$ does not depend on x. There exists further a measure ν which is equivalent to the Lebesgue measure μ and invariant with respect to the transformation $Tx=(\beta x)$, and for any measurable subset E of the interval $(0, 1)$ we have

$$\nu(E)=\int_E h(x)\,dx \tag{4.5}$$

where $h(x)$ is a measurable function and

$$1-\frac{1}{\beta}\leqq h(x)\leqq\frac{1}{1-\dfrac{1}{\beta}} \tag{4.6}$$

and we have

$$M(g)=\int_0^1 g(x)\,h(x)\,dx. \tag{4.7}$$

Proof. The β-expansion is an expansion with *dependent* digits. As a matter of fact, the admissible values for ε_n are $0, 1, \ldots, [\beta]$. But as

$$\sum_{n=1}^{\infty}\frac{[\beta]}{\beta^n}=\frac{[\beta]}{\beta-1}>1,$$

there exists a value N for which

$$\sum_{n=1}^{N}\frac{[\beta]}{\beta^n}>1.$$

This implies that the first N digits can not all be equal to $[\beta]$.

Thus not every sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ formed from the numbers $0, 1, \ldots, [\beta]$ is canonical. Let $S(n)$ denote the number of canonical sequences of order n for $n \geqq 1$ and put $S(0)=1$. Then $S(n)-S(n-1)$ is the number of those canonical sequences of order n for which $\varepsilon_n \neq 0$, because if $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1})$ is a canonical sequence of order $n-1$, then clearly $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}, 0)$ is a canonical sequence of order n, and conversely. In general, if $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}, \varepsilon_n)$ is a canonical sequence of order n, then $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1})$ is a canonical sequence of order $n-1$. Let us consider all canonical sequences $\mathcal{E}_{n-1}=(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1})$ of order $n-1$. If $(\varepsilon_1, \ldots, \varepsilon_{n-1}, k)$ is canonical for $k \leq k_{\mathcal{E}_{n-1}}$ but not for $k > k_{\mathcal{E}_{n-1}}$, then the intervals $\left[\frac{\varepsilon_1}{\beta}+\frac{\varepsilon_2}{\beta^2}+\cdots+\frac{\varepsilon_{n-1}}{\beta^{n-1}},\ \frac{\varepsilon_1}{\beta}+\frac{\varepsilon_2}{\beta^2}+\cdots+\frac{\varepsilon_{n-1}}{\beta^{n-1}}+\frac{k_{\mathcal{E}_{n-1}}}{\beta^n}\right)$ are clearly disjoint, and thus we have

$$\frac{1}{\beta^n}(S(n)-S(n-1))=\frac{1}{\beta^n}\sum k_{\mathcal{E}_{n-1}}\leqq 1,$$

consequently

$$S(n)-S(n-1)\leqq \beta^n \qquad (n=1,2,\ldots). \tag{4.8}$$

As $S(0)=1$, we obtain

$$S(n)\leqq \frac{\beta^{n+1}}{\beta-1} \qquad (n=1,2,\ldots). \tag{4.9}$$

Let us arrange the $S(n)$ numbers $\frac{\varepsilon_1}{\beta}+\frac{\varepsilon_2}{\beta^2}+\cdots+\frac{\varepsilon_n}{\beta^n}$, where $\mathcal{E}_n=(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ is a canonical sequence, and the number 1 according to their order of magnitude. Clearly the distance between any two consecutive terms does not exceed $\frac{1}{\beta^n}$. Thus we have

$$S(n)\geqq \beta^n. \tag{4.10}$$

From (4. 10) and (4. 9) we obtain incidentally

$$\lim_{n\to\infty} \sqrt[n]{S(n)} = \beta. \tag{4. 11}$$

Now let E denote any measurable subset of the interval (0, 1). As $T^{-n}E$ consists of $S(n)$ sets, each of which has a measure not exceeding $\frac{1}{\beta^n}\mu(E)$, we have

$$\mu(T^{-n}E) \leqq \frac{S(n)\mu(E)}{\beta^n} \leqq \frac{1}{1-\frac{1}{\beta}}\mu(E). \tag{4. 12}$$

On the other hand, $S(n)-S(n-1)$ of the sets mentioned above have the measure exactly equal to $\frac{\mu(E)}{\beta^n}$ and thus we obtain

$$\mu(T^{-n}E) \geqq \frac{(S(n)-S(n-1))\mu(E)}{\beta^n}. \tag{4. 13}$$

It follows by (4. 10) that

$$\frac{1}{n}\sum_{k=0}^{n-1}\mu(T^{-k}E) \geqq \frac{1}{n}\left(1+\sum_{k=1}^{n-1}\frac{(S(k)-S(k-1))}{\beta^k}\right)\mu(E) \geqq \left(1-\frac{1}{\beta}\right)\mu(E).$$

Thus we have

$$\left(1-\frac{1}{\beta}\right)\mu(E) \leqq \frac{1}{n}\sum_{k=0}^{n-1}\mu(T^{-k}E) \leqq \frac{1}{1-\frac{1}{\beta}}\mu(E). \tag{4. 14}$$

Applying again the theorem of DUNFORD and MILLER, Theorem 2 follows exactly in the same way as Theorem 1 in § 2. As regards the ergodicity of the transformation $Tx=(\beta x)$, it can be proved in the same way by using KNOPP's theorem as the ergodicity of the transformations $Tx=(f(x))$ considered in § 2. The only difference consists in that we choose now for J the class of those intervals $\left[\frac{\varepsilon_1}{\beta}+\frac{\varepsilon_2}{\beta^2}+\cdots+\frac{\varepsilon_n}{\beta^n}, \frac{\varepsilon_1}{\beta}+\frac{\varepsilon_2}{\beta^2}+\cdots+\frac{\varepsilon_n+1}{\beta^n}\right)$ for which not only the sequence $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ but also $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n+1)$ is canonical.

Let us consider an example.

EXAMPLE 4. Let us take $\beta=\frac{\sqrt{5}+1}{2}$ and put $\alpha=\frac{1}{\beta}=\frac{\sqrt{5}-1}{2}$. Then we have $\alpha+\alpha^2=1$. This implies that each digit $\varepsilon_n=1$ is followed by a digit

$\varepsilon_{n+1}=0$ and there does not exist any other dependence of the digits on each other.[9] This makes it easy to obtain in this special case a complete insight into the set of canonical sequences. It can be shown that in this case

$$h(x)=\begin{cases}\dfrac{5+3\sqrt{5}}{10} & \text{for} \quad 0\leqq x<\dfrac{\sqrt{5}-1}{2},\\[2ex] \dfrac{5+\sqrt{5}}{10} & \text{for} \quad \dfrac{\sqrt{5}-1}{2}<x\leqq 1,\end{cases}$$

and thus the limiting frequencies of the digits 0 and 1 are $\dfrac{5+\sqrt{5}}{10}$ and $\dfrac{5-\sqrt{5}}{10}$, respectively.

We hope to return to the explicit determination for an arbitrary $\beta>1$ of the measure which is invariant with respect to the transformation $Tx=(\beta x)$ and is equivalent to the Lebesgue measure (the proof of the existence of which is contained in Theorem 2) at another occasion.

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